Dynamic User Optimal Assignment with Physical Queues for a Many-to-Many OD Pattern

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1. Introduction

This research discusses the formulation and a solution algorithm of the dynamic user optimal assignment considering effects of physical queues with a given many-to-many OD pattern. In the dynamic user optimal assignment which is sometimes called the reactive assignment, vehicles are assumed to choose their routes based on present instantaneous travel times.

2. Network and Traffic Demand

A network consists of links and nodes. Sequential numbers from 1 to N are allocated to N nodes. The number of links is L and a link from node i to j is denoted as link (i,j). A time-dependent many-to-many OD demand is assumed to be given, which is denoted as

$Q_{ij}(t)$ = cumulative OD demand from origin i to destination node j generated at the origin by time t (given). (1)

The cumulative arrival and departure curves are defined as follows:

$A_{ij}(t)$ = the cumulative arrivals at link (i,j) by time t, (2)

$D_{ij}(t)$ = the cumulative departures from link (i,j) by time t. (3)

$\lambda_{ij}(t)$ = the arrival rate at link (i,j) at time t $= dA_{ij}(t)/dt$, (4)

$\mu_{ij}(t)$ = the departure rate from link (i,j) at time t $= dD_{ij}(t)/dt$. (5)

The $A_{ij}(t)$ and $\lambda_{ij}(t)$ are unknown variables which must be determined so as to establish the DUO assignment principle defined later.

3. Constraints

3.1. Flow Conservation at Nodes

The first constraint is the flow conservation at a node:

$$- \sum_{k} D_{ki}^{d}(t) + \sum_{j} A_{ij}^{d}(t) - Q_{id}(t) = 0 \text{ , } i = 1, 2, \ldots, N, \; i \neq d. \qquad (6)$$

$A_{ij}^{d}(t)$ = the cumulative arrivals at link (i,j) to destination d by time t,
$D_{ij}^{d}(t)$ = the cumulative departures from link (i,j) to destination d by time t.

3.2. First In First Out Discipline

Second, under the FIFO discipline, a vehicle must leave link (i,j) in the same order as its order of arrival at the link. Thus, the $A_{ij}(t)$ and $D_{ij}(t)$ must be related to each other through link travel time $T_{ij}(t)$:

$$A_{ij}(t) = D_{ij}(t + T_{ij}(t)) \quad \text{or} \quad A_{ij}^{d}(t) = D_{ij}^{d}(t + T_{ij}(t)) \qquad (7)$$

where $T_{ij}(t)$ = travel time on link (i,j) for a vehicle entering the link at time t. Therefore, the link travel time $T_{ij}(t)$ is written as a function of $A_{ij}(t)$ and $D_{ij}(t)$ [1, 2]:

$$T_{ij}(t) = D_{ij}^{-1}(A_{ij}(t)) - t \quad \text{or} \quad T_{ij}(t) = D_{ij}^{d\,-1}(A_{ij}^{d}(t)) - t. \qquad (8)$$

The derivative of (7) is obtained as

$$\lambda_{ij}^{d}(t) = \mu_{ij}^{d}(t + T_{ij}(t))\,(1 + dT_{ij}(t)/dt),$$

where $\lambda_{ij}^{d}(t) = dA_{ij}^{d}(t)/dt$ and $\mu_{ij}^{d}(t) = dD_{ij}^{d}(t)/dt$.

Since $\sum_{d} \mu_{ij}^{d}(t) = \mu_{ij}(t)$, eventually the departure rate becomes:

$$\mu_{ij}{}^{d}(t+T_{ij}(t))=\mu_{ij}(t+T_{ij}(t))\cdot\frac{\lambda_{ij}{}^{d}(t)}{\lambda_{ij}(t)} \quad . \qquad (9)$$

We clearly see the role of the FIFO discipline; that is, departure rate $\mu_{ij}{}^{d}(t + T_{ij}(t))$ is controlled not only by its arrival rate $\lambda_{ij}{}^{d}(t)$ but also by arrival rates to other destinations $\lambda_{ij}{}^{d'}(t)$'s, $d' \neq d$.

4. Physical Queues

Actual queues have some physical lengths. Once a link is fully occupied by a queue, the departure flow rate from the upstream link must be limited to the rate of the downstream link with the queue. To incorporate this phenomena, we have to first analyze the shock-wave speed of the congested flow and then discuss how the departure flow rate should be adjusted due to a queue downstream.

4.1. Shock-Wave Speed

To analyze the wave propagation, the following variables are introduced:

$F_{ij}(x,t)$ = the cumulative number of vehicles passing at location x on link (i,j) by time t,
$f_{ij}(x,t)$ = the flow rate at location x on link (i,j) at time t,
$k_{ij}(x,t)$ = the density at location x on link (i,j) at time t,

where location x means a length toward upstream from the downstream end on a link. By definition, the derivatives of flow and density are:

$$f_{ij}(x,t) = \partial F_{ij}(x,t)/\partial t\,, \qquad (10)$$

$$k_{ij}(x,t) = \partial F_{ij}(x,t)/\partial x \quad . \qquad (11)$$

Furthermore, the flow-density relationship is assumed a triangle shape as shown in Fig.1 at any time t and location x on link (i,j), in which the wave speed ($\partial f_{ij}/\partial k_{ij}$) is constant v_{ij} or v'_{ij}. From the flow conservation, we obtain

$$\partial k_{ij}(x,t)/\partial t = \partial f_{ij}(x,t)/\partial x. \qquad (12)$$

Thus, its derivative $df_{ij}(x,t)$ is written as follows:

$$
\begin{aligned}
df_{ij}(x,t) &= \partial f_{ij}(x,t)/\partial t \cdot dt + \partial f_{ij}(x,t)/\partial x \cdot dx \\
&= \{\partial f_{ij}(x,t)/\partial t + \partial f_{ij}(x,t)/\partial x \cdot dx/dt\} \cdot dt \\
&= \{\partial f_{ij}(x,t)/\partial t + \partial k_{ij}(x,t)/\partial t \cdot dx/dt\} \cdot dt \,.
\end{aligned}
$$

On a trajectory with speed of - $dx/dt = \partial f_{ij}/\partial k_{ij}$, $df_{ij}(x,t)$ hence becomes

$$
d\, f_{ij}(x,t) = \{\partial f_{ij}(x,t)/\partial t - \partial k_{ij}(x,t)/\partial t \cdot \partial f_{ij}(x,t)/\partial k_{ij}(x,t)\,\} \cdot dt = 0. \qquad (13)
$$

This means that flow $f_{ij}(x,t)$ does not change on the trajectory of the backward wave. With a triangle flow-density relationship, Newell [3] shows an interesting property in the cumulative curve $F_{ij}(x,t)$ in relation to flow $f_{ij}(x,t)$ and density $k_{ij}(x,t)$:

$$
\begin{aligned}
dF_{ij}(x,t) / dx &= \partial F_{ij}(x,t)/\partial x + \partial F_{ij}(x,t)/\partial t \cdot dt/dx \\
&= k_{ij}(x,t) + f_{ij}(x,t) \cdot dt/dx \\
&= k_{ij}(x,t) - f_{ij}(x,t) \cdot dk_{ij} / df_{ij} \\
&= k_{ij}(x,t) - f_{ij}(x,t)/ v'_{ij} \\
&= k^{max}_{ij}. \qquad (14)
\end{aligned}
$$

Since the above means that $dF_{ij}(x,t) / dx$ takes the same constant value of k^{max}_{ij} independent of location x, we can draw $F_{ij}(\ell_{ij}, t)$ at the upstream end of the link by shifting $D_{ij}(t) = F_{ij}(0,t)$ horizontally by $-\ell_{ij}/v'_{ij}$ and vertically by $k^{max}_{ij} \cdot \ell_{ij}$. In Figures 2 and 3, the shifted line $D'_{ij}(t)$ is shown and the intersection of $D'_{ij}(t)$ and $A_{ij}(t)$ is known as the shock wave. According to the theory, the lower line of $A_{ij}(t)$ or $D'_{ij}(t)$ shows the cumulative number of physical vehicles arriving at the upstream end of link (i,j), $F_{ij}(\ell_{ij}, t)$.

4.2. Adjustment of Departure Flow Rate

When a queue fully backs up on a link as for t_1 to t_2 in Figures 2 and 3, the departure flow rate from the upstream link must be limited up to the downstream link capacity. For point queues, the departure flow rate of vehicles leaving link (i,j) at $t + T_{ij}(t)$ (entering link (i,j) at time t) is defined using the constant link capacity μ_{ij}^* :

$$\mu_{ij}(t + T_{ij}(t)) = \begin{cases} \mu_{ij}^* \ , & T_{ij}(t) > \ell_{ij}/v_{ij} \ \text{ or } \ \lambda_{ij}(t) > \mu_{ij}^*, \\ \lambda_{ij}(t) \ , & \text{otherwise,} \end{cases} \tag{15}$$

If a vehicle is not delayed, it is assumed to travel on link (i,j) for minimum travel time ℓ_{ij}/v_{ij}. However, once link travel time $T_{ij}(t)$ gets larger than ℓ_{ij}/v_{ij} at time t or arrival rate $\lambda_{ij}(t)$ is larger than maximum departure rate μ_{ij}^*, departure rate $\mu_{ij}(t + T_{ij}(t))$ is assumed to be restricted to μ_{ij}^* due to a queue on the link. At present time t, since the link capacity is constant for point queues, the departure flow rate can be determined until time $t + T_{ij}(t)$.

On the other hand, for physical queues, the link capacity cannot be constant because it depends upon traffic condition downstream. However, the capacity would be determined based on downstream traffic condition by present time t (independent of future traffic condition). Therefore, in general, the departure flow rate $\mu_{ij}(t)$ can be written as a function of arrival rates at all links by time t:

$$\mu_{ij}(t) = \mu_{ij}(\boldsymbol{\lambda}(t') \mid t' < t) \tag{16}$$

where $\boldsymbol{\lambda}(t) = (\lambda_{11}(t), \lambda_{12}(t), \ldots\ldots\ldots\ldots, \lambda_{ij}(t), \ldots\ldots\ldots\ldots)^t$,

although the explicit functional form of $\mu_{ij}(\boldsymbol{\lambda}(t') \mid t' < t)$ depends on the road geometry as well as traffic condition at the downstream end of the link.

5. Dynamic User Optimal Assignment

5.1. Outline of the Assignment

Every vehicle is assumed to choose the shortest route to its destination at any time based on the present instantaneous link travel times. Let $\pi_{id}(t)$ be the shortest travel time from node i to destination d prevailing at time t, which means that $\pi_{id}(t)$ is the sum of link travel times along the shortest route p_{id} evaluated at time t: $\pi_{id}(t) = \sum_{(i,j)\in p_{id}} T_{ij}(t)$. Similar to the static assignment, the required condition for the DUO assignment is defined such that

$$\begin{cases} \pi_{id}(t) - \pi_{jd}(t) = T_{ij}(t), & \text{if a vehicle with destination } d \text{ leaving} \\ & \text{node } i \text{ at time } t \text{ uses link } (i,j), \\ \pi_{id}(t) - \pi_{jd}(t) \leqq T_{ij}(t), & \text{otherwise.} \end{cases} \quad (17)$$

According to the definition, the route choice of vehicles is clearly dependent only upon the instantaneous link travel times at present time t, but independent of the future link travel times. Therefore, the assignment is decomposed with respect to present time t; that is, we can consider the assignment sequentially from the beginning of the study time period.

At present time t, $\lambda_{ij}{}^{d}(t')$ and $\mu_{ij}{}^{d}(t')$ are assumed to be evaluated for every link and destination for $t' < t$. Let us consider to determine $\lambda_{ij}{}^{d}(t)$ for every link and destination. From (16), $\mu_{ij}(t)$ is first determined and departure rate by destination $\mu_{ij}{}^{d}(t')$ is determined from (9):

$$\mu_{ij}{}^{d}(t) = \mu_{ij}(t) \cdot \frac{\lambda_{ij}{}^{d}(\hat{t})}{\sum_{d'} \lambda_{ij}{}^{d'}(\hat{t})} \quad , \qquad t = \hat{t} + T_{ij}(\hat{t}) \quad .$$

Since the flow conservation must be satisfied at node i, the following result is obtained from(6):

$$\sum_{j} \lambda_{ij}{}^{d}(t) = q_{id}(t) + \sum_{k} \mu_{ki}{}^{d}(t), \quad (18)$$

where, $q_{id}(t) = dQ_{id}(t)/dt$ (given) $i = 1, 2, \ldots, N$, $i \neq d$.

The total arrival rate $\sum_{j} \lambda_{ij}{}^{d}(t)$ can be known because the right hand side of (18) has been evaluated but individual arrival rate $\lambda_{ij}{}^{d}(t)$ must be determined through the DUO assignment based on link travel times estimated below. Since the minimum of link travel time is ℓ_{ij}/v_{ij}, link travel times at time t is estimated using the departure flow rate at present time t :

$$\begin{aligned} T_{ij}(t) &= D_{ij}^{-1}(A_{ij}(t)) - t \,, \\ &\cong Max\, [\, \ell_{ij}/v_{ij} \,, \{ A_{ij}(t) - D_{ij}(t) \} / \mu_{ij}(t)]. \end{aligned} \quad (19)$$

Note that the estimated link travel time at time t may not be equal to the actual link travel time experienced by a vehicle entering the link at time t, but we use the above estimates for the assignment at present time t. Using the estimated link travel times $T_{ij}(s)$'s, the shortest route from node i to destination d can be determined without difficulty by a standard shortest route algorithm. Even if there are two or more equally shortest routes from node i to d, the total rate of $\sum_j \lambda_{ij}^d(t)$ could be loaded onto one of the shortest routes in order to establish the DUO assignment principle by definition. As the result, $\lambda_{ij}^d(t)$ for $\forall(i,j)$ and $\forall d$ is determined at time t.

5.2. A Solution Algorithm

First, the time axis is divided into small intervals of equal length Δt . The arrival and departure rates of link (i,j), $\lambda_{ij}^d(t)$ and $\mu_{ij}^d\ (t)$, are assumed constant during $[t\ ,\ t+\Delta t)$.

step 1: Initialize link flow rates, cumulative curves, link travel times and present time.

$\lambda_{ij}^d(t) := \lambda_{ij}^d(0),\quad t < 0,\quad \forall(i,j),\quad \forall d,$

$\mu_{ij}^d(t) := \mu_{ij}^d(0),\quad t < 0\ ,\quad \forall(i,j)\ ,\quad \forall d,$

$A_{ij}^d(t) := A_{ij}^d(0),\quad t \leq 0,\quad \forall(i,j)\ ,\quad \forall d,$

$D_{ij}^d(t) := D_{ij}^d(0),\quad t \leq 0,\quad \forall(i,j)\ ,\quad \forall d,$

$T_{ij}(t) := T_{ij}(0),\quad t \leq 0,\quad \forall(i,j)\ ,$

$t := 0\ .$

Set Δt as $\Delta t \leq \underset{(i,j)}{Min}\ \ell_{ij}/v_{ij}$.

step 2: Determine departure rates $\mu_{ij}(t)$ and $\mu_{ij}^d(t)$ from (16) and (9).

step 3: Estimate link travel time $T_{ij}(t)$ from (19).

step 4: Determine the total arrival rate at node i for $[t\ ,\ t+\Delta t)$, $\sum_j \lambda_{ij}^d(t)\cdot\Delta t$, based on (18).

step 5: Find the shortest path from node i to destination d based on the estimated link travel time $T_{ij}(t)$ and determine $\lambda_{ij}^d(t)\cdot\Delta t$ by loading $\sum_j \lambda_{ij}^d(t)\cdot\Delta t$ onto a link starting from node i on the shortest path as shown in Fig.4: if link (i,j) is on the shortest path, $\lambda_{ij}^d(t)\cdot\Delta t$:=

$\sum_j \lambda_{ij}^d(t)\cdot\Delta t$, otherwise zero.

step 6: Extend $A_{ij}^d(\cdot)$ and $D_{ij}^d(\cdot)$ from time t to $t+\Delta t$ by straight lines with slopes $\lambda_{ij}^d(t)$ and $\mu_{ij}^d(t)$ respectively as shown in Fig.4.

step 7: If $T_{ij}(t) > \ell_{ij}/v_{ij}$ (a queue exists on link (i,j)), extend $D'^d_{ij}(\cdot)$ by shifting $D_{ij}^d(t')$, $t < t' \leq t+\Delta t$, horizontally by $-\ell_{ij}/v'_{ij}$ and vertically by $k^{max}_{ij}\cdot\ell_{ij}$ as in Fig.4.

step 8: Update *Back-Up-Flag*$_{ij}$, which indicates whether a queue backs up to the upstream link: if $A_{ij}^d(t+\Delta t) \geq D'^d_{ij}(t+\Delta t)$, *Back-Up-Flag*$_{ij}$:= 1; otherwise zero.

step 9: Update present time as $t := t+\Delta t$ and return to step 2.

In step 1, the small time interval Δt is set such that $\Delta t \leq \underset{(i,j)}{Min}\ \ell_{ij}/v_{ij}$ because of the following reason. To determine departure rate in step 2, the backward wave generated at the downstream end of link (i,j) at time t should not reach the upstream end before $t+\Delta t$: $\Delta t \leq \underset{(i,j)}{Min} -\ell_{ij}/v'_{ij}$. If the wave reaches before $t+\Delta t$, departure rates of upstream links cannot be evaluated for $[t\ ,\ t+\Delta t)$. Also, Δt should not be larger than the link travel time because departure rate $\mu_{ij}^d(\tilde{t}\ )$ must be determined for $t \leq \tilde{t} < t+\Delta t$ based on $\lambda_{ij}^d(t')$、 $t' < t$: $\Delta t \leq \underset{(i,j)}{Min}\ \ell_{ij}/v_{ij}$. Normally, wave speed in free flow region v_{ij} is larger than $-v'_{ij}$, we obtain the constraint $\Delta t \leq \underset{(i,j)}{Min}\ \ell_{ij}/v_{ij} < \underset{(i,j)}{Min} -\ell_{ij}/v'_{ij}$.

In step 2, departure rate $\mu_{ij}(t)$ for $[t\ ,\ t+\Delta t)$ is determined from a function $\mu_{ij}(t)= \mu_{ij}(\lambda(t') \mid t' < t\ )$ specified in (16). As mentioned earlier, this function has to be designed so that it determine the departure rate both with and without queue backing up conditions referring to *Back-Up-Flag*$_{ij}$ evaluated in step 8.

In step 3, link travel time $T_{ij}(t)$ is estimated by extending $D_{ij}^d(\cdot)$ by a straight line with slope $\mu_{ij}^d(t)$ as shown in Fig.4. The estimated travel time is based on the departure rate during $[\ t,\ t+\Delta t\ )$, and hence the estimate may be different from the actual travel time experienced by a vehicle entering the link at time t. However, at present time t, a vehicle is assumed to choose the shortest path based on the estimated travel time.

In step 8, *Back-Up-Flag*$_{ij}$ is updated every Δt interval but cannot be revised during Δt. That is, even if $A_{ij}^{d}(t)$ and $D'^{d}_{ij}(\cdot)$ intersect each other during [t, $t+\Delta t$), *Back-Up-Flag*$_{ij}$ is updated at $t+\Delta t$. This causes an error in the solution, but we expect the error can be negligible if Δt is sufficiently small.

6. Summary and Future Research Needs

Given time dependent many-to-many OD volumes, we first discuss the formulation of the assignment so as to satisfy the flow conservation and the First-In-First-Out queue discipline. Then, defining the optimal condition, we extend the discussion with point queues to one with physical queues based on the kinematic wave theory by Newell.

References

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多起点多終点ＯＤにおける渋滞延伸を考慮した
動的利用者最適交通量配分

論文要旨

渋滞延伸の影響を考慮した多起点多終点ＯＤの動的利用者最適配分問題について、定式化と解法を提案する。動的利用者最適配分とは、 利用者が現時点までの交通状況に基づいて、終点までの最短経路を選択すると仮定した配分である。起点からの出発時刻別に多起点多終点ＯＤは与えられているものとし、交通量の保存則、First-In-First-Out 原則を満たす制約条件を整理した後、動的利用者最適条件を定式化して、本配分問題が現在時刻によって分解できることを示した。次に物理的な長さを持つ待ち行列を考え、ショックウェーブ理論より、渋滞が上流に延伸して先詰まりを起こす現象をモデルに組み込んだ。

Abstract

This research proposes the formulation and a solution algorithm for the dynamic user optimal assignment with physical queues.　A many-to-many OD pattern is assumed to be given, which means users' departure time choices have been determined.　Under the DUO assignment, users are assumed to choose their shortest routes to their destinations based upon current instantaneous travel time, but not based on their actual travel time (or cost).　First, the flow conservation and the First-In-First-Out discipline are established as constraints of the assignment.　Second, the optimality condition of the assignment is formulated.　Third, based on the shock-wave theory by Newell, the back-up phenomena of physical queues are included in the model.

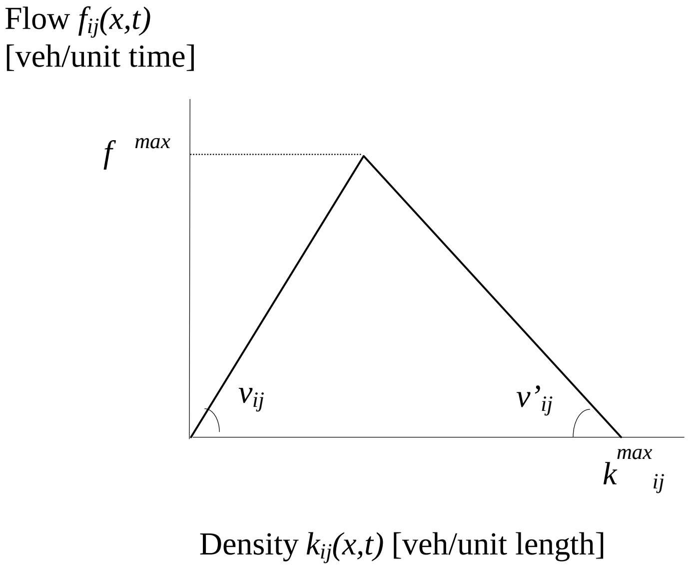

Fig. 1 A Flow - Density Relationship on link *(i,j)*

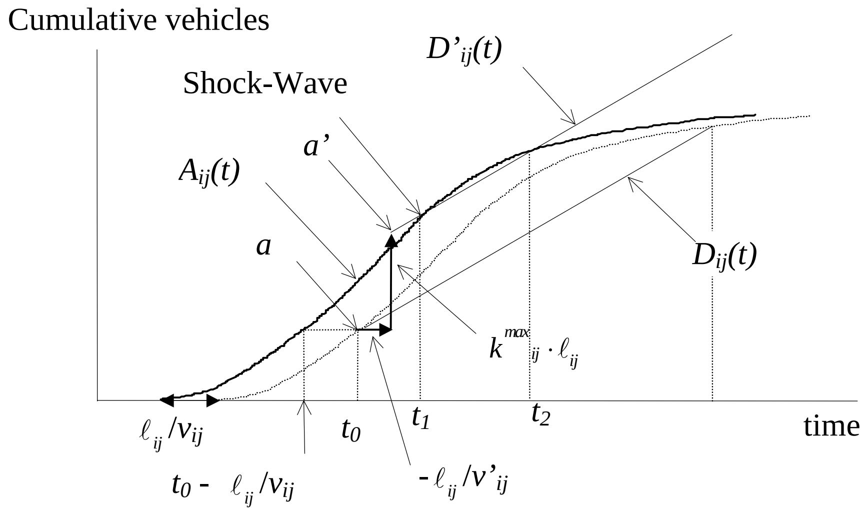

Fig. 2 Backward Wave Propagation and Vehicle Trajectories on link *(i,j)*

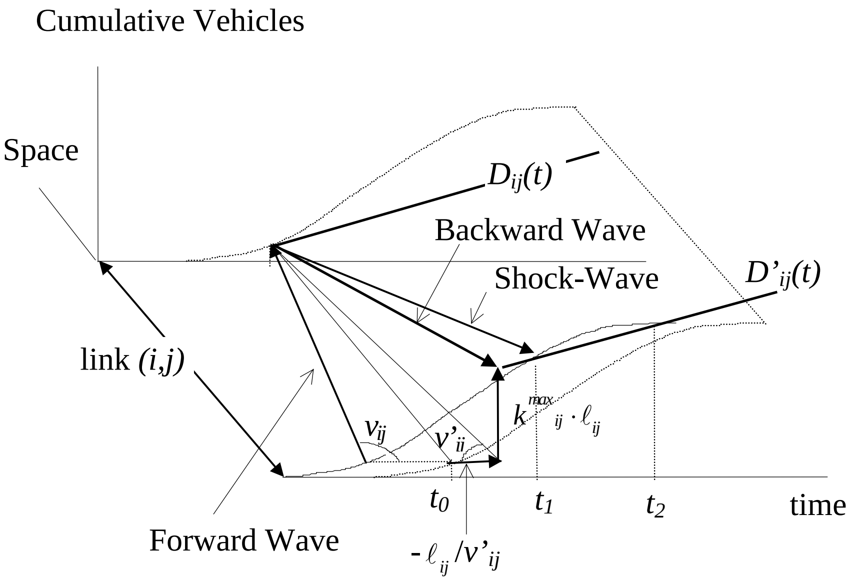

Fig.3 A three-dimensional Illustration of Backward Wave Propagation on link *(i,j)*

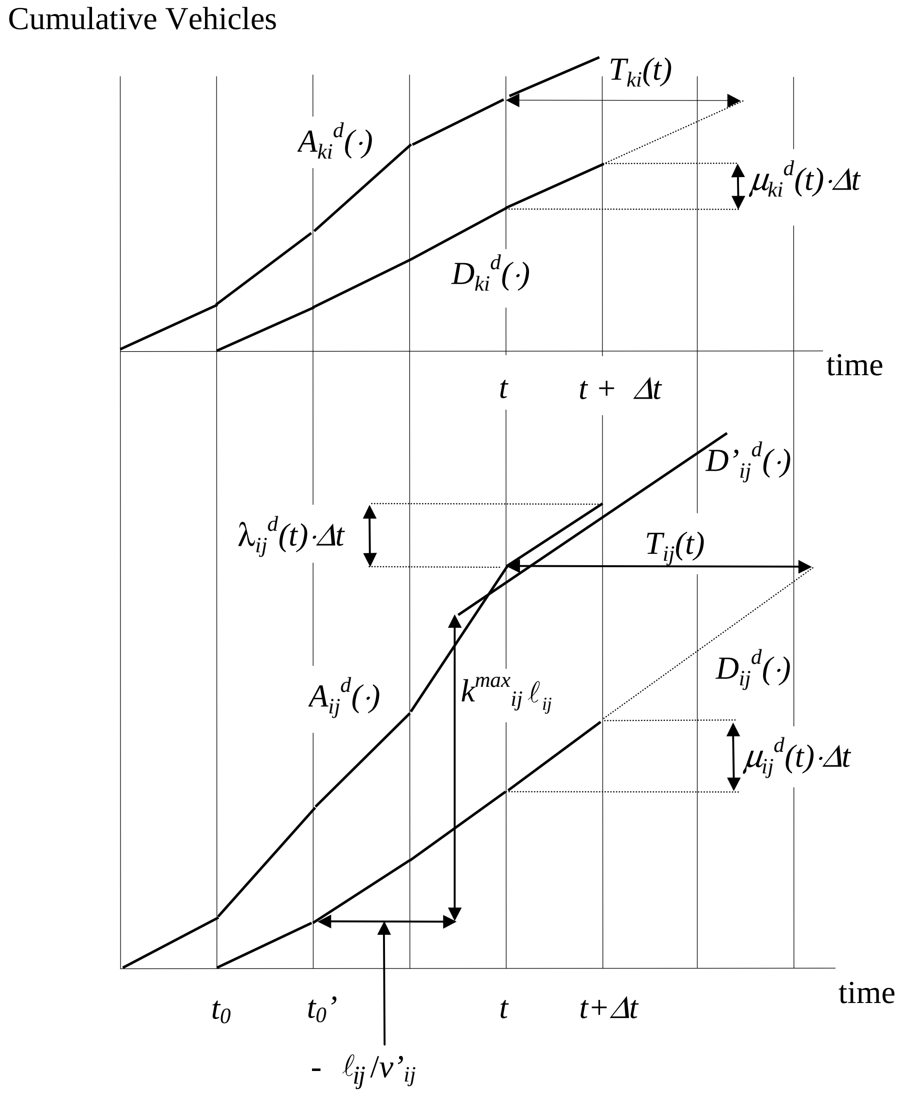

Fig. 4 Construction of Cumulative Arrival and Departure Curves on link *(i,j)*