OPTIMAL TOLL PATTERN ON A ROAD NETWORK UNDER STOCHASTIC USER EQUILIBRIUM WITH ELASTIC DEMAND

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1. INTRODUCTION

For an efficient use of roads, many economists have suggested to impose tolls on road users based on the marginal cost principle. Also, in the studies of network assignment, the marginal cost principle has been theoretically discussed for an achievement of the System Optimum flow pattern(e.g.,1,2,3,4).

These researches are based on deterministic user equilibrium (UE) theory where all the users are assumed to have the perfect knowledge on travel time. The assumption of the perfect knowledge is , however, somewhat unrealistic, because road users may sometimes have only the partial knowledge and the knowledge could also contain some noises. To incorporate these uncertainty and randomness to the network assignment, the Stochastic User Equilibrium assignment (SUE) has been proposed. The concept of stochastic user equilibrium is based on random utility theory and the network assignment models with fixed demand have been discussed by Daganzo and Sheffi(5), Fisk(6), and Daganzo(7).

The authors have discussed the road pricing policy under the Stochastic User Equilibrium in our previous paper(8). This paper extends the former research to the elastic demand situation (SUE/ED) where the congestion tolls may cause not only route choice but also the OD demand itself to shift. Depending on travel costs, some users apparently switch routes between origins and destinations, but sometimes they may also give up trips or change trip ends, so the OD demand would change also.

We first formulate the optimal toll problem under SUE/ED as a mathematical programming. The object of the problem is to maximize social surplus, which is the difference between the user surplus and the total travel time in the network. The optimality condition of the problem is next shown as an extended version of the marginal cost principle in the deterministic assignment. The algorithm which computes the optimal link tolls without path enumeration is proposed and some numerical examples are provided.

2. FORMULATION

This chapter introduces the optimal congestion toll problem as a mathematical programming problem. Following two sections formulate the user behavior model and the objective function of the optimal toll problem respectively. TABLE 1 shows the notation used here.

TABLE 1 Notations

$\Delta = [\,\delta^{rs}_{ap}\,]$	the link path incidence matrix, where $\delta^{rs}_{ap} = \begin{cases} 1 & \text{if link a is on path p of OD-pair rs,} \\ 0 & \text{otherwise.} \end{cases}$
x_a	the flow on link a
t_a	the travel time on link a
$\hat{t}_a$	the marginal travel time on link a
e_a	the congestion toll on link a
c_a	the general cost on link a
X^{rs}_p	the flow on path p from origin r to destination s
C^{rs}_p	the general cost on path p from origin r to destination s
T^{rs}_p	the travel time on path p from origin r to destination s
$\hat{T}^{rs}_p$	the marginal travel time on path p from origin r to destination s
E^{rs}_p	the congestion toll on path p from origin r to destination s
Q_{rs}	the flow from origin r to destination s
S_{rs}	the expected perceived minimum travel cost from origin r to destination s

2.1 User Behavior Model

We assume that not only the path choice pattern but also the OD demand are described by the Stochastic User Equilibrium with Elastic Demand (SUE/ED) model, which is the generalization of User Equilibrium (UE) model.

The SUE/ED model consists of three parts, that is, the relation among variables in network, path choice model, and OD demand model.

(a) Relation among basic variables

Using link-path incidence matrix Δ, link flow $\mathbf{x} = [\,x_1, .., x_a, ..]^t$ can be represented as the sum of path flow $X = [..\, X^{rs}_1, .., X^{rs}_p, ..]^t$:

$$\mathbf{x} = \Delta \;\; X \tag{1}$$

Path travel time $T = [..\, T^{rs}_1, .., T^{rs}_p, ..]^t$ can be calculated by summing up the link travel time $t = [\,t_1, .., t_a, ..]^t$ on the path :

$$T = \Delta^t t \; . \tag{2}$$

Similarly, we adopt the toll system satisfying the following conditions between path toll $E = [..\, E^{rs}_1, .., E^{rs}_p, ..]^t$ and link toll $e = [\,e_1, .., e_a, ..]^t$:

$$E = \Delta^t e \; . \tag{3}$$

We define general link cost $c = [c_1, .., c_a, ..]^t$ consisting of link travel time t and link toll e as follows:

$$c = t + e \ . \tag{4}$$

Also, we define general path cost C by the manner similar to (2),

$$C = \Delta^t c \ . \tag{5}$$

(b) Path choice model

It is assumed that a user chooses a path so as to minimize his own path cost which varies stochastically as shown below:

$$\tilde{C}_p^{rs} = C_p^{rs} + \varepsilon_p^{rs}, \tag{6}$$

where ε is the stochastic term which means the randomness of perceived cost. According to the random utility theory, the path choice probability of path p for each OD-pair (r,s) is given by following equation:

$$P_p^{rs} = \text{Prob}(\tilde{C}_p^{rs} \leqq \tilde{C}_{p'}^{rs} \ ; p \neq p') \tag{7}$$

Multiplying OD flow Q with path choice probability P, we can get path flow,

$$X_p^{rs} = Q_{rs} P_p^{rs} \ . \tag{8}$$

(c) Demand model

For a short range demand analysis, the OD demand could be fixed and given, but not for a long range analysis. Considering the increase of congestion and the flex time system, for example, users may choose times of trip making based on their travel costs between OD pairs. Hence, it is natural to assume that the OD demand for each time period itself is a function of the travel cost between the OD pair.

We assume a nested structure for trip making - path choice behavior. Then, we can assume that trip making behaviors are determined by the following expected perceived minimum cost based on random utility theory(9,10).

$$S_{rs} = E\,[\min_p.\{\tilde{C}_p^{rs}\}]\ , \tag{9}$$

where, $E\,[\cdot]$ denotes the operation of expectation and the OD demand is given by the following separable demand function:

$$Q_{rs} = Q_{rs}(S_{rs})\ . \tag{10}$$

2.2 Objective Function

Our objective of imposing tolls is to maximize the difference between the social benefit and cost, though we may define many other objectives in practice. This can be formulated as the following mathematical programming problem which is widely known as the System Optimum assignment (we call it SO) problem with separable demand function.

$$\text{(O1)} \quad \min. Z^{SO}(\mathbf{x}, \mathbf{Q}) = \sum_a x_a t_a(\mathbf{x}) - \sum_{rs} \int_0^{Q_{rs}} Q_{rs}^{-1}(\omega)\, d\omega \tag{11}$$

The first and second term represents the sum of travel time and all the user surplus in the network respectively.

Though $\mathbf{x}$ and $\mathbf{Q}$ are explicit control variables in SO problem, we can not control the flow variables directly in our problem, since ($\mathbf{x}$, $\mathbf{Q}$) are determined automatically through the user behavior model mentioned in previous section. Instead of direct control of ($\mathbf{x}$, $\mathbf{Q}$), we intend to control them indirectly by imposing proper congestion tolls on users. This paper considers the following two strategies as a toll charging system; ① collect toll $\mathbf{e}$ on link basis and ② collect toll $\mathbf{E}$ on path basis,though various systems may be possible in accordance with the characteristics of considering problem. Note that tolls are not included into the social costs because the collected tolls can be redistributed to increase the social benefit by many ways such as constructing new facilities, improving link capacities, and so on.

This problem can be also viewed as a game where network manager(player1) seeks to maximize social surplus by imposing tolls based on user behavior; on the other hand, network users(player2) behave so as to attain individual optimality under the conditions given by the manager. This is so called the Stackelberg problem whose objective function of the upper level problem is (O1) and of the lower level problem is SUE/ED model.

3. ANALYSIS

3.1 The Case of Symmetric Link Cost Jacobian

(1) Equivalent optimization problem of user behavior model

In general, the SUE/ED model can be solved from the simultaneous equations of (1),(5)-(10). However, we convert the optimal toll problem to equivalent mathematical programming problems. Here, we particularly employ the link travel time as a function of only that link flow:

$$t_a = t_a(x_a). \tag{12}$$

If the Jacobian of the link cost function is asymmetric, we can not get the following equivalent optimization problem as discussed later.

For the logit based path choice model, the equivalent optimization problem is

(P1)

$$\min. Z^{SUE}(\mathbf{x}, \mathbf{Q}) = \sum_a \int_0^{x_a} c_a(\omega)\, d\omega - \sum_{rs} Q_{rs} H_{rs} - \sum_{rs} \int_0^{Q_{rs}} Q_{rs}^{-1}(\omega)\, d\omega, \tag{13}$$

subject to

(1) and (5),

$$Q_{rs} = \sum_p X_p^{rs}, \tag{14}$$

$$Q_{rs} \geqq 0 \quad , \quad x_a \geqq 0 \; , \tag{15),(16}$$

where

$$H_{rs} = -\frac{1}{\theta} \sum_p (X_p^{rs} / Q_{rs}) \ln (X_p^{rs} / Q_{rs}) , \tag{17}$$

θ = a parameter depending on a variance of perceived cost.

The dual problem of (P1) gives useful information on properties of optimization problem. In particular, we use the dual problem to develop the algorithm for the solution later in Chapter 5. We can obtain the following dual of (P1):

(D1)

$$max.\ Z_D^{SUE} (c) = - \sum_a \int_{c_{a0}}^{c_a} c_a^{-1} (\nu) \, d\nu + \sum_{rs} \int_{S_{rs0}}^{S_{rs}} Q_{rs} (\nu) \, d\nu \tag{18}$$

subject to

$$c_a \geqq c_{a0} \equiv c_a (0) \tag{19}$$

where,

$$S_{rs} = -\frac{1}{\theta} \ln \sum_p \exp(- \theta C_p^{rs}) \tag{20}$$

The equivalency of these programming problems to the SUE/ED model can be shown from the Kuhn-Tucker condition. Also, the solution is uniquely determined, since the objective functions of both problems are strictly convex with respect to unknown variables and the constraints are convex sets.

(2) Optimality condition

Suppose that the objective function of SUE/ED (P1) coincides with that of SO:

$$Z^{SUE} [\mathbf{x} (c), Q (c)] = Z^{SO} [\mathbf{x} (t), Q (t)] \tag{21}$$

Then, it is clear that the SUE flow pattern depending on general travel cost is, at the same time, the system optimum, since both of the problem O1 and P1 have the same constraints for ($\mathbf{x}$, Q). If we could find the toll pattern satisfying Eq.(21), the toll pattern would be globally optimum. By transforming the total travel time into an integral form using marginal travel time $\hat{t}$, Eq.(21) is simplified to

$$\sum_{rs} Q_{rs} H_{rs} = \sum_a \int_0^{x_a} c_a (\omega) \, d\omega - \sum_a \int_0^{x_a} \hat{t}_a (\omega) \, d\omega \tag{22}$$

Taking the partial derivative of Eq.(22) with respect to P, we obtain the optimality condition for path variables.

$$\partial H_{rs} / \partial P_p^{rs} = C_p^{rs} - \hat{T}_{rs} \tag{23}$$

In addition, since the OD demand obtained from the SUE/ED model must also coincide with that of SO, the following condition is required from the definition of the demand function:

$$S_{rs}\,[\,T^{\bullet}_{rs} + E_{rs}\,] = \hat{T}^{\bullet}_{rs}\ . \tag{24}$$

where $\hat{T}^{\bullet}_{rs}$ is the marginal path cost of OD pair rs in the SO state.

Since expected perceived minimum path cost is viewed as the generalization of equilibrium minimum path cost in Wardropian user equilibrium model, we can interpret that the optimality condition above, Eq.(24), is an extension of the conventional marginal cost principle:

$$\min_{p}.\{\, T^{rs}_{p} + E^{rs}_{p} \} = \hat{T}^{\bullet}_{rs} \tag{25}$$

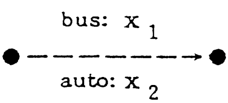
by which one can maximize social surplus in deterministic Wardropian UE model. Note that Eq.(24) is both the part of necessary and sufficient conditions for optimal toll in the SUE/ED model, though it is not necessary in the fixed demand SUE.

3.2 The Case of Asymmetric Link Cost Jacobian

As seen in the previous studies, the SUE model can not be the equivalent optimization problem when link cost function is asymmetric. However, if the Jacobian of link cost function is positive definite, equilibrium flow pattern is uniquely determined and the same optimality conditions as mentioned earlier would be obtained.

If the Jacobian is not positive definite , the equilibrium pattern can not be uniquely determined and it is difficult to analyse general properties of the equilibrium pattern. However, interesting phenomena concerning optimal toll problem can be found in the following simple network.

Let's consider a network which has 1 OD pair with given total demand D and 1 path with 2 modes of bus and auto. The bus and auto are denoted by subscripts 1 and 2. Flows of two modes are denoted by x_1 and x_2, and the corresponding travel costs are c_1 and c_2.



$$x_1 + x_2 = D \quad = \text{fixed} \tag{26}$$

The cost functions of the bus and auto are assumed to mutually influence each other. For simplicity, we assume that this interaction is affin, that is ,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} A_{10} \\ A_{20} \end{bmatrix}. \tag{27}$$

Also, we consider the case of $A_{21} \neq A_{11}$, $A_{22} = A_{12}$, and then cost difference between the bus and auto becomes a function depending only on bus flow.

$$c_2 - c_1 = (A_{21} - A_{11}) x1 + (A_{20} - A_{10}) \quad (28)$$

Suppose that user behavior is represented by logit model:

$$x_1 = D \exp(-\theta c_1) / \{\exp(-\theta c_1) + \exp(-\theta c_2)\} \quad (29)$$

By defining $c \equiv c_2 - c_1$, $x \equiv x_1$, $\alpha \equiv 1/(A_{21} - A_{11})$, $\beta \equiv (A_{10} - A_{20})\alpha$, we can get the following Demand-Supply equilibrium problem.

$$\text{Supply} : c = (x - \beta) / \alpha \quad (30)$$

$$\text{Demand} : x = D / \{1 + \exp(-\theta c)\} \quad (31)$$

This SUE Problem has multiple equilibrium depending on parameter α, β, and θ. For intuitive consideration of this phenomena, depict the demand-supply curves on $x - c$ plain as in Fig.1. It is easily seen that the points that two curves intersect are equilibrium solutions. For $0 \leqq \alpha \leqq \theta D / 4$, demand and supply curves have two or three equilibrium points.

Figure 2 depicts a relation between the equilibrium solution x and parameter β for $0 \leqq \alpha \leqq \theta D / 4$, where $(x1, x2)$ and $(\beta 1, \beta 2)$ are:

$$x1 = \frac{D}{2}\{1 - \sqrt{1 - \frac{4}{D}\frac{\alpha}{\theta}}\}, \quad \beta 1 = x1 + \frac{\alpha}{\theta} \ln \frac{x2}{x1}, \quad (32),(33)$$

$$x2 = \frac{D}{2}\{1 + \sqrt{1 - \frac{4}{D}\frac{\alpha}{\theta}}\}, \quad \beta 2 = x2 + \frac{\alpha}{\theta} \ln \frac{x1}{x2}. \quad (34),(35)$$

This figure shows very strange phenomena arising in our toll problem. Suppose the initial value of β is sufficiently large, then equilibrium flow would be almost zero. Also, consider that β decreased to the level of $\beta 2$ as the result of proper congestion toll imposed on auto (this means the increase of A_{20}). Then, bus demand will slightly increase to the level of $x2'$. This slight increase of bus demand will continue within the limit of $\beta > \beta 1$. Further increase of the toll will cause the sudden increase of bus demand jumping from $x1$ to $x1'$ at $\beta = \beta 1$. Conversely, suppose the toll is imposed on bus. Then, the increase causes the contrary phenomena; that is, bus demand will decrease smoothly in the range $\beta < \beta 2$ and then will decrease suddenly jumping from $x2$ to $x2'$ at $\beta = \beta 2$.

The change of α have also influence on the dynamics of x, though we consider the dynamics of equilibrium flow x solely corresponding to the change of parameter β fixing parameter α so far. Considering the change of (α, β) simultaneously, the relation between equilibrium flow and the parameters can be illustrated as Figure 3. Projecting this curved surface to $\alpha - \beta$ plain, we have the curve with a cusp as is illustrated in Figure 4.

Furthermore, the equilibrium flow has also interesting dynamics corresponding to the change of demand side parameter θ. The dynamics on $\theta - x$ plain may be classified by the range of parameter α, β like Figure 5.

It is seen from Figure 5 that there are multiple equilibrium points when the parameter set (α, β, θ) satisfies $\alpha > 0$, $0 < \beta < D$ and $\theta > \theta c$, where $\theta c = \alpha D / \{(D - x) x\}$ and x is the solution of

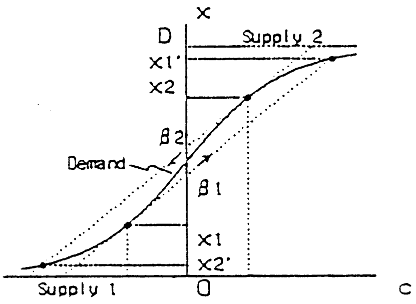


Fig. 1 Equilibrium of Demand-Supply functions.

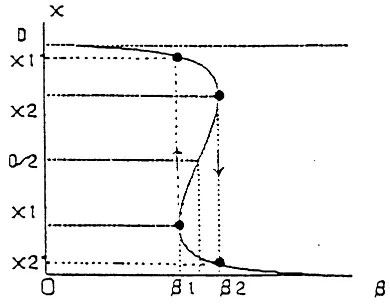


Fig. 2 Equilibrium flow corresponding to Parameter β.

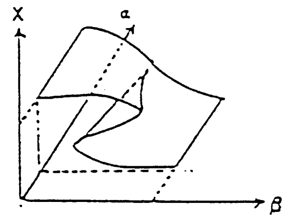


Fig. 3 Equilibrium flow and parameter (α, β).

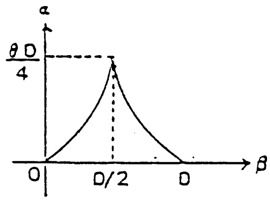


Fig. 4 Projection of equilibrium surface to $\alpha - \beta$ plain.

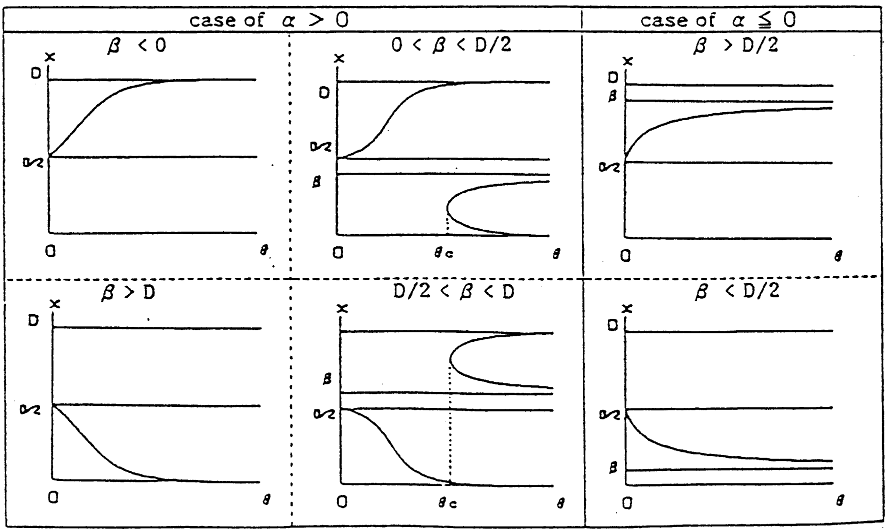


Fig.5 Equilibrium flow and parameter θ

$$\ln \frac{x}{x - D} = \frac{D}{(D - x)\,x} \quad . \tag{36}$$

These phenomena mentioned above are concerning with bifurcation theory in mathematics and we imagine that deriving optimal toll in general for asymmetric link cost Jacobian is almost impossible. Thus, the following chapters concentrate on the discussions on the derivation of the link and path tolls, particularly for the case with symmetric link cost Jacobian.

4. OPTIMAL TOLL PATTERNS

4.1 Optimal Path Toll

This section derives the optimal path toll for the logit based SUE/ED model with the symmetric link cost Jacobian. In the description below, superscript $^{\bullet}$ denotes the variables established at the SO state.

Separating toll from cost in Eq.(23), we can get the following formula for path toll E.

$$E_p^{rs} = \hat{T}^{\bullet}_{rs} - T_p^{rs\bullet} + \left. \frac{\partial H_{rs}}{\partial P_p^{rs}} \right|_{P = P^{\bullet}} \tag{37}$$

The $P^{\bullet}$, $\hat{T}^{\bullet}$, $T^{\bullet}$ in this equation are given variables which can be calculated from the SO flow pattern. For other general random utility model, H in Eq.(37) may be replaced by the conjugate function of expected perceived minimum cost function S, which can not be expressed as a explicit function of P in general, and so the optimal toll can not be expressed explicitly. Only for the logit model, however, we can obtain the path toll analytically. From Eq.(17) and Eq.(37), we get the following optimal path toll,

$$E_p^{rs} = \hat{T}^{\bullet}_{rs} - T_p^{rs\bullet} - \frac{1}{\theta}\{\ln P_p^{rs\bullet} + 1\} \, . \tag{38}$$

The first and second term in this equation is a marginal travel time same as the congestion toll in the deterministic UE, and the third additive term comes from the stochastic choice behavior. This equation means that a path which has a low choice probability in the SO flow; i.e. the path not desirable to be used to attain the SO flow, should be plausibly imposed a large toll. Furthermore, if value of θ in Eq.(38) were to increase to infinity ($\theta \to \infty$) —— this is equivalent to letting the variance of perceived travel cost be 0; that is, the flow pattern becomes the deterministic user equilibrium —— then the third term of the right hand side would be eliminated, which is consistent with the conventional congestion toll theory.

4.2 Optimal Link Toll

This section discusses the uniqueness of optimal link toll $e^{\bullet}$ which is consistent with optimal path toll $E^{\bullet}$. In the discussion below, we will not distinct OD pairs but paths by serial number, and we use notation K, L, N for the number of all paths, links, and nodes excluding OD nodes respectively. Also, we assume $K \geqq L \geqq N$ (real networks often satisfy this condition).

From the discussion on the optimality condition in the previous chapter, the SUE path flow pattern should coincide with the SO path flow pattern X^*. However, the SO path flow pattern X^* can not be determined uniquely from SO link flow pattern x^* in general. On the other hand, the SUE path flow pattern is unique. Thus, only one path flow pattern should be selected from the multiple SO path flow patterns to attain the SUE state at the same time.

Since path toll vector E with K elements corresponding to path flow X^* can be derived from (37), optimal path toll E^* is determined uniquely.

On the other hand, path toll E and link toll e have thefollowing relationship:

$$E = \Delta^t e \ , \tag{39}$$

where Δ is a $L \times K$ matrix whose (a,p) element is δ_{ap}.

The rank of incidence matrix Δ is $L - N$ because of the definition of δ_{ap} and flow conservation at each node. Hence, Δ can be transformed to the following matrix A using proper $L \times L$ matrix V and $K \times K$ matrix W.

$$A \equiv V \Delta W = \begin{array}{cl} \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] & \begin{array}{l} L-N \\ N \end{array} \\ \begin{array}{cc} L-N & K-(L-N) \end{array} & \end{array} , \tag{40}$$

where I is a $(L-N) \times (L-N)$ unit matrix.

Multiplying W^t to Eq.(39) from the left, we have

$$\tilde{E} = A^t \tilde{e} \ , \tag{41}$$

$$\text{where} \quad \tilde{E} \equiv W^t E \ , \tag{42}$$

$$\tilde{e} \equiv (V^t)^{-1} e. \tag{43}$$

From Eq.(41) and (42), L-N elements of $\tilde{e}$ are determined from $(E_1, .., E_{L-N})$ and the remaining N elements are arbitrary constant (i.e. independent of e). From Eq.(43),

$$\tilde{e} = V e \ . \tag{44}$$

Thus, L-N elements of e are determined by $(E_1, .., E_{L-N})$ and remaining N elements are arbitrary constant μ which is independent of E.

$$
e = \begin{bmatrix} e_1(E_1, \ldots, E_{L-N}, \mu) \\ \vdots \\ e_{L-N}(E_1, \ldots, E_{L-N}, \mu) \\ \mu_1 \\ \vdots \\ \mu_N \end{bmatrix} \tag{45}
$$

From (37) and (45), the N elements of optimal link toll e^* are arbitrary constant μ independent of the SO flow pattern and the remaining L-N elements become functions of μ. The remaining L-N elements could be uniquely determined in accordance with the SO link flow pattern, x^* and μ.

$$
e^* = \begin{bmatrix} e_1(x^*, \mu) \\ \vdots \\ e_{L-N}(x^*, \mu) \\ \mu_1 \\ \vdots \\ \mu_N \end{bmatrix} \tag{46}
$$

In order to obtain link toll vector e from path toll vector E, we should give N link tolls μ in advance, and the remaining L-N elements in e are uniquely determined from L-N independent equations (45). We may give this N elements of vector μ from various practical constraints. For example, if we impose tolls on zone bases, the link tolls would be constrained so that a user pays a uniform fixed charge within each of the zones.

5. ALGORITHM FOR GENERAL NETWORK

The preceding discussions are concentrated on the analytical derivation of the optimal path toll and the uniqueness conditions of the link toll. Using these results, we can calculate both path and link tolls, if path enumeration is possible. Since path enumeration is not , however, feasible in practical network, it is needed to develop the algorithm for computing the optimal link toll in a general network without path enumeration. We present an algorithm for the logit based SUE/ED model with the symmetric link cost Jacobian below. The algorithm is based on the fact that minimum expected cost S can be calculated from path choice entropy H which can be decomposed into other entropy calculatable from link flows.

Since the optimal link toll can attain the conventional SO flow pattern from the discussions of previous chapters, the following mathematical programming problem D1' is yielded from D1, where the inverse link cost function in Eq.(18) is replaced with SO link flow x^*.

$$
\text{(D1')} \quad \mathit{min.}\ Z[c(e)] = -\sum_{rs} D_{rs}(S_{rs}) + \sum_a x_a^* c_a(t_a^*, e_a), \tag{47}
$$

where

$$
D_{rs}(w) = \int_0^w Q_{rs}(\nu)\, d\nu .
$$

To solve problem D1', we adopt the subgradient algorithm which has been successfully applied to network assignment problem(11,12). To apply this algorithm, we need a value of the (sub)gradient , the objective function and its lower bound at a current link toll pattern.

From the property of S, the derivative of Z with respect to e becomes as follows(10):

$$d_a \equiv \partial Z / \partial e_a = \sum_{rs} \sum_{p} Q_{rs}(S_{rs}) P_p^{rs} \delta_{ap}^{rs} - x_a^{*} \equiv y_a - x_a^{*}. \tag{48}$$

Since y_a represents flow on link a determined by the logit based path choice model, the y can be calculated without path enumeration for for any given S by Dial's Algorithm(13), the first term . Also, from Eq.(47), we see that the value of S must be calculated without path enumeration to obtain the objective value.

According to the conjugate theory, expected perceived minimum cost S and path choice entropy H are satisfying the following relationship in the logit based path choice model (14):

$$S_{rs} + H_{rs} = \sum_{p} P_p^{rs} C_p^{rs} = \sum_{a} y_a^{rs} c_a / Q_{rs} \tag{49}$$

where y^{rs} is the link flow by OD pair.

Furthermore, path choice entropy H can be decomposed into a form consisting of only link flows, if path choice probability P is determined by the logit formula (15).

$$H_{rs} = -\frac{1}{\theta} \sum_{ij} p_{ij}^{rs} \ln p_{ij}^{rs} + \frac{1}{\theta} \sum_{i} \{ (\sum_{j} p_{ij}^{rs}) \ln (\sum_{j} p_{ij}^{rs}) \}, \tag{50}$$

where $p_{ij}^{rs} = y_a^{rs} / Q_{rs}$, $\begin{cases} i = \text{a starting node of link a,} \\ j = \text{a terminal node of link a.} \end{cases}$

If we give any values of link costs assuming the link toll e, the link flow by OD-pair y_{rs} can be computed from Dial's algorithm. Therefore, we can determine path choice entropy H_{rs} from Eq.(50), which determines S_{rs} from Eq.(49) without path enumeration.

Using Eq.(49) and the optimality condition (24), the lower bound can be calculated easily. The first term of objective function at optimal can be evaluated by substituting marginal travel time into $D(\cdot)$, since Eq.(24) hold at the optimal solution. The second term can be evaluated by summing up marginal travel time and path choice entropy, since Eq.(49) hold at optimal. Hence, lower bound LB can be evaluated by

$$LB = -\sum_{rs} D_{rs}(\hat{T}_{rs}^{*}) + \sum_{rs} (\hat{T}_{rs}^{*} + H_{rs}) Q_{rs}^{*}. \tag{51}$$

Thus, we can summarize the algorithm for calculating the optimal toll as follows.

PHASE 1 (Solve the System Optimum assignment) :

Calculate optimal OD flow Q^*, marginal travel time $\hat{T}$, link flow x^*, and link travel time t^*.

PHASE 2 (Solve problem D1' by the subgradient algorithm) :

Step 0: Select real numbers LB and ε, and a starting point $e^1 \geqq e^0$, where ε is a parameter of the convergence criterion. Set $w = \infty$ and $n=1$.

Step 1: ① Calculate link flow y^n corresponding to $c^n(e^n)$, using Dial's algorithm.

② Calculate the value of path choice entropy $H^n(c^n)$ from Eq.(50).

③ Calculate the value of expected minimum cost $S^n(c^n)$ from Eq.(49).

Step 2: Calculate $Z(c^n)$ and gradient $d^n(c^n)$ from Eq.(48).

Step 3: Update LB using Eq.(51).

Step 4: If $Z(c^n) < w$, set $w = Z(c^n)$ and go to Step 5; otherwise, go to Step 6.

Step 5: If $w - LB \leqq \varepsilon \mid LB \mid$, terminate as e^n is ε-optimal; otherwise, go to Step 6.

Step 6: Update link toll by $e^{n+1} = \max\{e^0, e^n - \alpha^n d^n\}$,

where the max is taken component-wise and the step size α^n is determined by

$$\alpha^n = \lambda^n\{Z(c^n) - LB\}/\| d^n \|^2, \quad 0 < \lambda^n < 2.$$

Return to Step 1.

6. NUMERICAL EXAMPLE

6.1 An Example of Unique Link Toll Pattern

The network illustrated in Figure 6 has three nodes and five links. The link cost functions have the form,

$$t_a(x_a) = A_a x_a^4 + B_a,$$

and TABLE 2 shows the value of the parameters. This network includes two OD pairs W1: ①→③ and W2: ①→②. W1 has 6 paths and W2 has 2 paths. The OD flows are determined by the demand functions,

$$Q_i(S_i) = \exp(-\alpha S_i + \beta), \quad i \text{ denotes the OD pair,}$$

where the value of the parameters are α = [0.2, 0.2] and β = [1.0, 0.5]. The SO flow pattern (x^*, Q^*) and corresponding travel time pattern (t^*, T^*) are shown in TABLE 3 (these are easily calculated using the Frank-Wolfe algorithm).

The number of links L=5 and number of nodes excluding OD nodes N=0 yield the rank of the incidence matrix L−N=5, which is the same as the number of links. Hence, the optimal link toll pattern is determined uniquely. Applying the algorithm mentioned in Chapter 5, we can obtain the optimal link toll e = [3.672, 3.356, 1.441, 1.403, 1.357] for θ = 1.0.

One can verify numerically that SUE flow pattern under this link toll pattern is just the same with SO flow pattern ($\mathbf{x}^*$, $\mathbf{Q}^*$).

6.2 An Example of Non-Unique Link Toll Pattern

The network structure, link cost function, demand function, and the parameters are the same as the above, but now we exclude OD pair W2. The SO flow pattern ($\mathbf{x}^*$, $\mathbf{Q}^*$) and corresponding travel time pattern ($\mathbf{t}^*$, $\mathbf{T}^*$) are shown in TABLE 4. We can not determine the link toll pattern uniquely for this case, because the number of links L=5 and number of nodes excluding OD nodes N=1 yield the rank of the incidence matrix L−N=4, which is less than the number of links. Implementing our algorithm, one can verify non-uniqueness of the toll pattern numerically. At first, let all the initial values of link tolls be 10. Then the optimal link tolls become $\mathbf{e}$ = [0.386, 0.090, 3.533, 3.495, 3.449]. Next, fix the toll of link 1 at 1.0, and the corresponding link tolls become $\mathbf{e}$ = [1.000, 0.706, 2.917, 2.878, 2.833]. Though these toll patterns are apparently different, both types of tolls can attain the SO flow pattern.

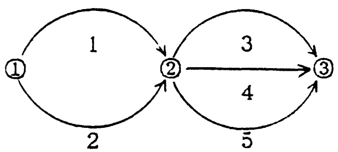


Fig.6 Network for numerical example

TABLE 2 Link cost function parameters

link	1	2	3	4	5
A a	2.0	1.0	7.0	6.0	5.0
B a	0.6	0.8	0.5	0.5	0.5

TABLE 3 SO flow in example of unique link toll pattern.

OD		$\hat{T}$ i	Q i
1		5.086	0.983
2		4.245	0.705
link	t a	$\hat{t}$ a	x a
1	1.329	4.245	0.777
2	1.489	4.245	0.911
3	0.568	0.841	0.314
4	0.568	0.841	0.327
5	0.568	0.841	0.342

TABLE 4 SO flow in example of non-unique link toll pattern.

OD		$\hat{T}$ i	Q i
1		3.623	1.317
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link	t a	$\hat{t}$ a	x a
1	0.884	2.022	0.614
2	1.044	2.022	0.703
3	0.720	1.601	0.421
4	0.720	1.601	0.438
5	0.720	1.601	0.458

7. CONCLUDING REMARKS

We considered the optimal toll problem under stochastic user behavior with elastic demand (SUE/ED) as an extension of the conventional congestion toll theory. We have shown that the optimal toll problem is formulated in as mathematical programming problem which maximizes the social surplus defined as the difference between the user surplus and total travel costs. We employ the logit based SUE/ED model and show that the optimality condition of the problem can be explained as an extended version of the marginal cost principle suggeted in the deterministic theory.

We have discussed two types of toll collecting policies: link and path tolls policies in which tolls are imposed on each of network links and on paths in the network respectively. For the logit based SUE/ED model, the optimal path tolls can be expressed in a closed form. The link tolls can be also evaluated from the path toll; however, we need L - N additional requirements to uniquely determine the link tolls, where L is the number of links and N is the number of nodes excluding the OD node. In practice, it is difficult to impose a different toll on each of links, but common tolls may be collected from links in a certain area. We could use these practical requirements for the additional requirements to determine the link tolls. Finally, an algorithm for the optimal link toll calculation is proposed, in which the path enumeration is not required.

The problem considered here would not only be one of the theory in transportation economics but also be viewed as a kind of the sensitivity analysis of network equilibrium model or the optimal network design problem. Also, it may be possible to apply the method to the problem of estimating parameters of the link cost function from observed link flows.

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