TRANSPORTATION

AND

TRAFFIC THEORY

Proceedings of the 12th International Symposium on
the Theory of Traffic Flow and Transportation
Berkeley, California, USA, 21-23 July, 1993

A symposium in honor of Gordon F. Newell

edited by

CARLOS F. DAGANZO
Department of Civil Engineering
University of California
Berkeley, CA, USA

1993
ELSEVIER
AMSTERDAM • LONDON • NEW YORK • TOKYO

Transportation and Traffic Theory / C.F. Daganzo (Editor)

Dynamic Equilibrium Assignment with Queues for a One-to-Many OD Pattern

Masao Kuwahara[a] **and Takashi Akamatsu**[b]

[a] Institute of Industrial Science, University of Tokyo, 7-22-1 Roppongi, Minato-ku, Tokyo 106, Japan
[b] Nomura Research Institute Ltd., 9-1, Nihonbashi 1-Chome, Chuo-ku, Tokyo 103, Japan

Abstract

This research deals with the dynamic traffic assignment on an over saturated network for a one-to-many origin-destination pattern under the user equilibrium principle. The purpose of this study is to obtain time dependent cumulative arrival curves at each of the nodes explicitly taking into account the effects of queues given time dependent OD volumes; that is, the departure times of users are assumed to be known. We first show that the problem can be decomposed with respect to the starting time from an origin. It is then shown that the problem forms a loop structure, which is formulated as an extension of the static assignment using two kinds of unknowns, link flows and arrival times at nodes. The solution is proved to exist, provided that the link travel time is monotone increasing in the link flow. However, a unique solution is not generally guaranteed. An algorithm to obtain the solution is finally proposed and applied to a simple network.

1 Introduction

This research deals with the dynamic traffic assignment on an over saturated network with queues under the user equilibrium principle. The time dependent origin-destination demand is a one-to-many pattern which is assumed given; that is, only route choices are considered in this paper but departure time choices from an origin are given.

Several researchers have studied the dynamic traffic assignment on the discrete demand space and networks consisting of links and nodes. Merchant et. al. [1, 2] formulated the dynamic flow model under the system optimal principle, given a one-to-many OD demand. The formulation by Merchant et. al. was then further investigated by Ho [3] and Carey [4]. Friesz et. al. [5] also considered this type of model and extended the model so that it could dealt with a many-to-many OD pattern. Wie et. al. [6] and Boyce et. al. [7] studied the similar models under the user optimal principle instead, where they assumed that users chose the fastest routes to destinations based on the instantaneous information on link travel times. Most of these studies assumed that the link travel time was a function of the number of vehicles on the link, which was evaluated using the so called exit function. The exit function determined the departure flow rate as a function of the number of vehicles existing on the link as well. Although they developed mathematically interesting models, there was no clear connection between the exit function they introduced and real queuing phenomena. Also, we should note that the user optimal principle proposed by Wie and Boyce does not establish the usual user equilibrium where no one can find a better route than one he is currently assigned.

Recently, Smith et. al. [8,9] studied the dynamic assignment under the user equilibrium principle. They considered the existence of queues on a network more explicitly and disclosed several different characteristics of the dynamic assignment from the static one. However, the formulation of the assignment is still in a primitive stage.

On the other hand, the different type of queuing problems for the morning and evening commute trips have been studied on a continuum demand space with a many-to-one OD pattern [10 - 17]. Given commuters' work schedules, their departure times as well as routes were determined so as to establish an equilibrium. Although they considered queuing delay (waiting time in a queue) explicitly, the study has been so far restricted to a network with a limited number of bottlenecks. In particular, every commuter was assumed to pass a bottleneck only once in most studies in this category.

This research can be considered as an extension of the above queuing analyses to more general networks possibly having many queues, but we deal with a discrete demand space and consider only route choices similar to the evening commute problem.

2 An Outline of the Study

A network consists of links and nodes, and the given time dependent OD demand is assumed to be generated from a single origin and absorbed in various destination nodes. Every user is assumed to choose a route so as to minimize his travel time which consists of the static free flow travel time and time dependent queuing delay.

Figure 1 shows an example of the cumulative arrival and departure curves on link (i,j) where all the variables are defined with respect to the arrival time at node i, t_i. The trip cost of a vehicle entering link (i,j) at time t_i is shown as $T_{ij}(t_i)$. Our objective is to determine cumulative arrival curves $A_{ij}(t)$ for every link (i,j) for all time t so as to establish an equilibrium.

Under the user equilibrium state, the order of vehicle arrivals at any node can be shown to be the same as the order in which those vehicles depart from the single origin. For this reason, the arrival time at node i, t_i, is related to departure time from the single origin s in an equilibrium condition and all the variables in Fig.1 can be redefined with respect to time s.

Suppose that a vehicle which leaves the origin at time s enters link (i,j) at time $t_i(s)$. The vehicle must spend travel time $T_{ij}(t_i(s))$ on the link, but under FIFO (*First In First Out*), its travel time essentially depends only on the arrival curve $A_{ij}(t)$ before the arrival time $t_i(s)$ but independent of $A_{ij}(t)$ thereafter. Thus, together with the above discussion on the order of arrivals, it is concluded that the vehicle's travel time depends only on route choices of others leaving the single origin before its departure time s.

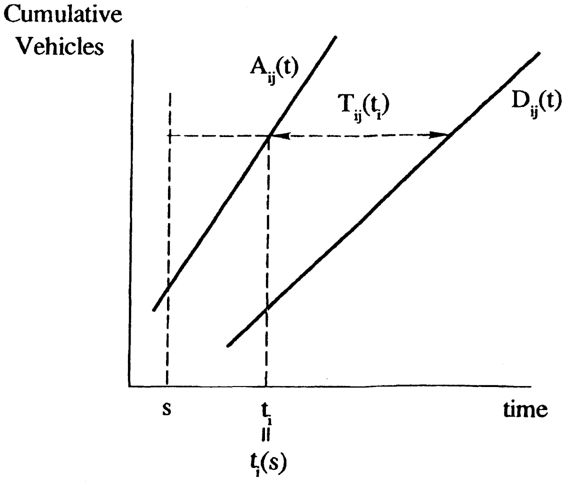

Fig.1 Cumulative Arrivals and Departures on link (i,j)

The equilibrium assignment can be, therefore, decomposed so that we consider the equilibrium sequentially in the order of departures from the single origin. For instance, the time axis is divided into small intervals, and we start considering the equilibrium route choices only for vehicles leaving the origin during the first time interval. Then, the equilibrium in the next interval should be analyzed given the route choices up to the previous interval, and so on.

3 Dynamic Network Flows and Link Travel Time with Queues

3.1 Network and Traffic Demand

A network consists of links and nodes. Sequential numbers from 1 to N are allocated to N nodes. The number of links is L and a link from node i to j is denoted as link (i,j).

A time-dependent one-to-many OD demand is assumed to be given, which is denoted as

$Q_{rj}(s)$ = cumulative OD demand from the single origin r to destination node j generated at the origin by time s (given).

Let us also introduce demand functions with respect to the arrival time at a node, t.

$$
\begin{aligned}
R_r(t) &= \text{cumulative trips generated at single origin r by time t,} \\
&= \textstyle\sum_j Q_{rj}(t), \\
S_j(t) &= \text{cumulative trips absorbed at destination node j by time t.}
\end{aligned}
$$

Here, s means the starting time from the single origin and t represents the arrival time at a node. Since, at the origin, clearly starting time s is equal to the arrival time at origin r, $R_r(t)$ can be known from the given $Q_{rj}(s)$. However, in general, arrival time at node $j(j \neq r)$ can not be known in advance. Thus, $S_j(t)$ can be normally evaluated only through the dynamic assignment.

3.2 Cumulative Functions and Link Travel Time

The cumulative arrival and departure curves are defined as follows:

$$
\begin{aligned}
A_{ij}(t) &= \text{the cumulative arrivals at link (i,j) by time t,} \\
D_{ij}(t) &= \text{the cumulative departures from link (i,j) by time t,} \\
\lambda_{ij}(t) &= \text{the arrival rate at link (i,j) at time t} = dA_{ij}(t)/dt, \\
\mu_{ij}(t) &= \text{the departure rate from link (i,j) at time t} = dD_{ij}(t)/dt, \\
\mu^*_{ij} &= \text{the maximum departure rate from link (i,j), (given).}
\end{aligned}
$$

The travel time on link (i,j) at time t consists of a static free flow travel time m_{ij} and time dependent queuing delay $w_{ij}(t)$ as shown in (1) and Fig.2. Although it is possible to introduce costs of travel rather than the real travel time as the conventional traffic assignment, the link travel time is not here converted to monetary term to eliminate further complication. The queuing delay is evaluated based on the point queue concept in which a queue has no physical length and the FIFO queue discipline. In other words, a queue is assumed to form vertically at the end of each link. Thus, as shown in Fig.2, $Tij(t)$ is evaluated as the horizontal distance between $A_{ij}(t)$ and $D_{ij}(t)$ at arrival time t. And apparently, once the arrival curve, $A_{ij}(t)$, were known by time t, $T_{ij}(t)$ could be determined; that is, $T_{ij}(t)$ depends only on arrivals by time t.

Based on the above assumptions, the queuing delay of a vehicle entering link (i,j) at time t, $w_{ij}(t)$, is described using the number of vehicles in the queue, $X_{ij}(t)$, which consists of vehicles entering link (i,j) by time t :

$$X_{ij}(t) = A_{ij}(t) - D_{ij}(t + m_{ij}) \geq 0.$$

Since a queue is assumed to vertically form at the end of the link, $X_{ij}(t)$ is the vertical distance between the cumulative arrival curve at time t and the departure curve at time $t + m_{ij}$ as shown in Fig.2. Consequently, the queuing delay can be written as follows:

$$w_{ij}(t) = w_{ij}(X_{ij}(t)) = X_{ij}(t)/\mu^*_{ij} = (A_{ij}(t) - D_{ij}(t + m_{ij}))/\mu^*_{ij} \geq 0.$$

As a whole, travel time $T_{ij}(t)$ is summarized also as a function of $X_{ij}(t)$:

$$\begin{aligned} T_{ij}(t) = T_{ij}(X_{ij}(t)) &= m_{ij} + w_{ij}(X_{ij}(t)) \\ &= m_{ij} + X_{ij}(t)/\mu^*_{ij} \geq m_{ij}, \end{aligned} \tag{1}$$

m_{ij} = static free flow travel time on link (i,j), which is given,
$X_{ij}(t)$ = $A_{ij}(t)$ - $D_{ij}(t+m_{ij})$,
$w_{ij}(t)$ = $w_{ij}(X_{ij}(t))$,
= time dependent queuing delay for a vehicle entering link (i,j) at time t.

Compared to the previous studies, many of them assumed that link travel time, $T_{ij}(t)$, was a function of the number of vehicles existing on the link, which is almost equivalent to $X_{ij}(t)$ defined here. Then, the exit function was introduced to determine the departure flow rate, $\mu_{ij}(t)$, as a function of $X_{ij}(t)$ as well. However, this research evaluates the departure flow rate based on the deterministic queuing theory under the FIFO discipline, instead.

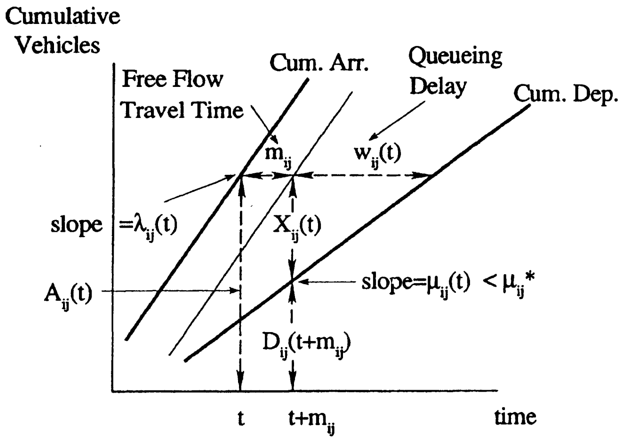

Fig.2 Cumulative Curves and Link Travel Time

Using the link travel time, the flow conservation on link (i,j) is written as:

$$A_{ij}(t) = D_{ij}(t + T_{ij}(t)). \tag{2}$$

This equation means that a vehicle entering link (i,j) at time t reaches node j at the later time by the travel time of $T_{ij}(t)$ under FIFO. Although the flow conservation on a link is related to its travel time $T_{ij}(t)$ in this way, this property has been neglected in most of the previous studies.

On the other hand, the flow conservation at node j is written as

$$-\sum_i D_{ij}(t) + \sum_k A_{jk}(t) - R_j(t) + S_j(t) = 0, \quad j = 1, 2, ..., N. \tag{3}$$

in which $R_j(t) = 0$ if j is not equal to single origin r by the definition. The first and last terms give the cumulative number of vehicles flowing into node j by time t, while the other terms describe the number of vehicles leaving node j by t.

4 Dynamic User Equilibrium Assignment

4.1 Definition of the User Equilibrium

Every vehicle is assumed to choose the route so as to minimize the travel time to its destination. Then, in general, the user equilibrium is defined as a condition where no vehicle can find a faster route than one it is currently assigned. Let $\tau_i(s)$ be the earliest arrival time at node i for a vehicle leaving the origin at time s. In other words, time interval $\tau_i(s) - s$ is the fastest travel time from the origin to node i. Under the user equilibrium state, in order for link (i,j) to be used by a vehicle leaving the origin at s, the link must be on the fastest route. Similar to the static assignment, the equilibrium condition is defined such that

$$\begin{cases} \tau_j(s) - \tau_i(s) = T_{ij}(\tau_i(s)), & \text{(if a vehicle starting the origin at s uses link (i,j))}, \\ \tau_j(s) - \tau_i(s) \leq T_{ij}(\tau_i(s)), & \text{(otherwise)}. \end{cases} \tag{4}$$

This condition means that if a vehicle starting from the origin at s uses link (i,j), the vehicle must enter the link at $\tau_i(s)$, spend $T_{ij}(\tau_i(s))$ on the link, and reach node j at $\tau_i(s) + T_{ij}(\tau_i(s))$ which must be equal to the earliest arrival time of $\tau_j(s)$.

4.2 Decomposition of the assignment by starting times from the origin

Let us first consider the order of arrivals at a node. In general, under the equilibrium state, a vehicle departing from one particular origin earlier must arrive at any node earlier than the others leaving the same origin later than the vehicle. Suppose that two vehicles 1 and 2 travel two routes 1 and 2 respectively from the same origin to a node. Vehicle 1 departs from the origin at time s_1 and vehicle 2 departs at time s_2 ($> s_1$), and they respectively take routes 1 and 2, and reach the node at time t_1 and t_2. If vehicle 2 arrives at the node earlier than vehicle 1 ($t_2 < t_1$), this is not the equilibrium state. The reason is that, in this situation, if vehicle 1 took route 2 instead of route 1, it could arrive earlier than vehicle 2 because of $s_1 < s_2$ under the FIFO discipline. Within vehicles leaving the same origin, this property must be met even for a many-to-many OD pattern.

However, here we have only one origin. Therefore, under the equilibrium state, the order of arrival at any node must be the same as the order of departure from the single origin. And, by the definition of the equilibrium, the arrival time at node i must be equal to the earliest time $\tau_i(s)$, which is related to the departure time from the origin, s.

As defined in section 3.2, link travel time $T_{ij}(\tau_i(s))$ depends only on cumulative arrivals and departures before time $\tau_i(s)$. Therefore, together with the above discussion on the order of arrivals at a node, it is concluded that $T_{ij}(\tau_i(s))$ depends only on route choices of those leaving the single origin before time s. Consequently, we can consider the equilibrium assignment sequentially in the order of departures from the single origin.

4.3 Formulation of the Equilibrium Assignment

Based on the above discussion on the order of arrivals at a node, let us formulate the dynamic equilibrium assignment. First, since the arrival time at link (i,j), t, is related to the starting time from the origin, s, such that t = $\tau_i(s)$, the travel time defined in (1), $T_{ij}(t) = T_{ij}(X_{ij}(t))$, is written as

$$T_{ij}(\tau_i(s)) = T_{ij}(X_{ij}(\tau_i(s))).$$

Second, according to the definition of the equilibrium, (4), if link (i,j) is used by vehicles leaving the origin at time s, $\tau_j(s)$ must be equal to $\tau_i(s) + T_{ij}(\tau_i(s))$. Thus, flow conservation (2) becomes

$$A_{ij}(\tau_i(s)) = D_{ij}(\tau_j(s)).$$

Applying this relationship to (3), we obtain the flow conservation in the following way eliminating D_{ij}:

$$-\sum_i A_{ij}(\tau_i(s)) + \sum_k A_{jk}(\tau_j(s)) - R_j(\tau_j(s)) + S_j(\tau_j(s)) = 0. \tag{5}$$

We should notice that now the OD demand of not only R_j but also S_j can be known as $S_j(\tau_j(s)) = Q_{rj}(s)$, since only vehicles starting from the origin before time s can reach node j by time $\tau_j(s)$. This flow conservation must be satisfied at each node j, j = 1, 2, ..., N ; however, only $(N-1)$ of them are independent each other because the sum of S_j over node j must be equal to R_j. This flow conservation is also described using the arrival flow rate by taking derivative with respect to starting time s:

$$\begin{aligned} -\sum_i dA_{ij}(\tau_i(s))/ds &+ \sum_k dA_{jk}(\tau_j(s))/ds \\ &- dR_j(\tau_j(s))/ds + dS_j(\tau_j(s))/ds = 0, \end{aligned} \tag{6}$$

in which $dA_{ij}(\tau_i(s))/ds$ means the rate of arrivals at link (i,j) for vehicles leaving the single origin at time s. This flow conservation (6) as well as (4) are the required conditions to establish the dynamic equilibrium.

In order to solve the problem, we however discretize the starting time. Let δs be a fixed unit interval of the starting time from the origin. In the previous section, we have concluded that the assignment problem can be decomposed regarding the starting time from the single origin. Let us therefore consider only vehicles leaving the origin during an interval $[s - \delta s, s]$ assuming that the equilibrium flow pattern of vehicles leaving the origin before time $s - \delta s$ has been obtained (thick lines in Fig.3).

For flow conservation (6), if we integrate each term over a time period of [s-δs, s], the following simple description is yielded:

$$-\sum_i y_{ij} + \sum_k y_{jk} - \delta R_j + \delta S_j = 0, \tag{7}$$

where

$$\begin{aligned}
y_{ij} &= \int_{s-\delta s}^{s} dA_{jk}(\tau_j(s))/ds \cdot ds = A_{ij}(\tau_i(s)) - A_{ij}(\tau_i(s-\delta s)), \\
\delta R_j &= \int_{s-\delta s}^{s} dR_j(\tau_j(s))/ds \cdot ds = R_j(\tau_j(s)) - R_j(\tau_j(s-\delta s)), \\
&= \sum_k Q_{jk}(s) - \sum_k Q_{jk}(s-\delta s), \\
\delta S_j &= \int_{s-\delta s}^{s} dS_j(\tau_j(s))/ds \cdot ds = S_j(\tau_j(s)) - S_j(\tau_j(s-\delta s)), \\
&= Q_{rj}(s) - Q_{rj}(s-\delta s).
\end{aligned}$$

The y_{ij} can be considered as the number of vehicles entering link (i,j) during the period of [s - δs, s] as shown in Fig.3. Also since the number of vehicles in a queue on link (i,j), $X_{ij}(\tau_i(s))$, is approximated as

$$X_{ij}(\tau_i(s)) \simeq A_{ij}(\tau_i(s-\delta s)) + y_{ij} - D_{ij}(\tau_i(s) + m_{ij}),$$

travel time $T_{ij}(X_{ij}(\tau_i(s)))$ defined in (1) is rewritten as follows:

$$\begin{aligned}
T_{ij}(X_{ij}(\tau_i(s))) &= m_{ij} + X_{ij}(\tau_i(s))/\mu^*_{ij} \\
&\simeq m_{ij} + (A_{ij}(\tau_i(s-\delta s)) + y_{ij} - D_{ij}(\tau_i(s) + m_{ij}))/\mu^*_{ij} \\
&= T_{ij}(\tau_i(s-\delta s)) + y_{ij}/\mu^*_{ij} - (\tau_i(s) - \tau_i(s-\delta s)) \geq m_{ij}.
\end{aligned}$$

Travel time of link (i,j) is now described as a function of y_{ij} and $\tau_i(s)$, $T_{ij}(y_{ij}, \tau_i(s))$, since unknowns are only them in the above equation but $T_{ij}(\tau_i(s-\delta s))$ and $\tau_i(s-\delta s)$ have been evaluated from thick lines in Fig.3.

And the equilibrium condition (4) becomes

$$\begin{cases} \tau_j(s) - \tau_i(s) = T_{ij}(y_{ij}, \tau_i(s)) & (\text{if} \quad y_{ij} > 0) \\ \tau_j(s) - \tau_i(s) \leq T_{ij}(y_{ij}, \tau_i(s)) & (\text{if} \quad y_{ij} = 0). \end{cases} \tag{8}$$

In this formulation, unknowns are y_{ij}'s of all L links and $\tau_i(s)$'s at all nodes except the origin, but both δR_j and δS_j are known from the given OD demand, Q_{jk}'s, as explained above. In Fig.3, if y_{ij} and $\tau_i(s)$ were obtained, arrival curve $A_{ij}(t)$ could be extended to time $\tau_i(s)$, and consequently departure curve $D_{ij}(t)$ could be also drawn to time $\tau_j(s)$ based on the deterministic queuing theory. Usually, the static assignment determines only link flows, which correspond to y_{ij}'s here; however, the dynamic problem contains one more dimension of $\tau_i(s)$'s.

Out of the two kinds of unknowns, if $\tau_i(s)$ were known, the problem becomes exactly the same as the well known static assignment:

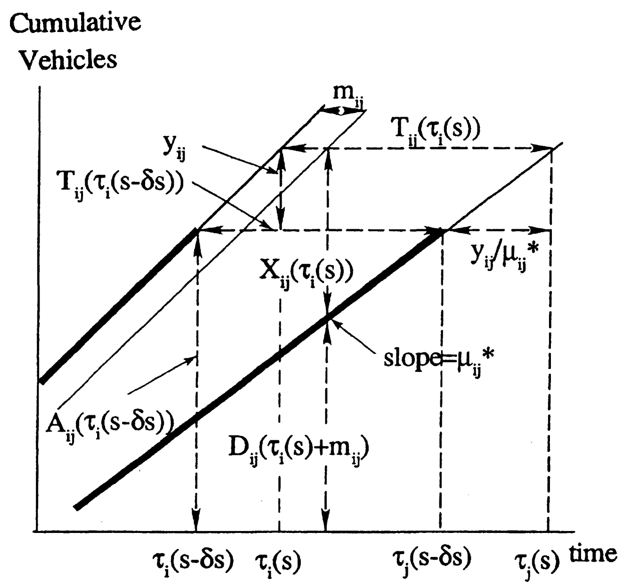

Fig.3 Travel Time on link (i,j) for a vehicle starting from the origin at time s

$$
\begin{aligned}
P(\tau): \min_{\mathbf{y}} F(\mathbf{y},\tau) &= \min_{\mathbf{y}} \sum_{ij} \int_0^{y_{ij}} T_{ij}(\omega, \tau_i) d\omega, \\
s.t. \quad h_j(\mathbf{y}) &= -\sum_i y_{ij} + \sum_k y_{jk} - \delta R_j + \delta S_j = 0, \qquad for\ all\ j, \\
g_{ij}(y_{ij}) &= -y_{ij} \le 0, \qquad for\ all\ (i,j).
\end{aligned}
$$

where

$$
\begin{aligned}
\tau_i &= \tau_i(s) \\
\tau &= \text{a column vector of } \tau_i\text{'s with (N-1) elements} = (\tau_1, ..., \tau_i, ..., \tau_{N-1})^t \\
\mathbf{y} &= \text{a column vector of } y_{ij}\text{'s with L elements} = (y_{12}, ..., y_{ij}, ...)^t
\end{aligned}
$$

The y_{ij}'s are determined so as to minimize the objective function. The Lagrangean of the optimization problem is

$$
L(\mathbf{y}, \tau, \eta, \phi) = F(\mathbf{y}, \tau) + \sum_j \eta_j h_j(\mathbf{y}) + \sum_{ij} \phi_{ij} g_{ij}(y_{ij}), \tag{9}
$$

where Lagrangean multipliers η and ϕ are column vectors of η_i's with (N-1) elements and ϕ_{ij}'s with L elements respectively. The Kuhn-Tucker condition is generally described as:

$$
\partial L / \partial y_{ij} = T_{ij}(y_{ij}, \tau_i) - \eta_j + \eta_i - \phi_{ij} = 0 \tag{10}
$$

$$
\begin{aligned}
h_j(\mathbf{y}) &= 0 && (11)\\
\phi_{ij} g_{ij}(y_{ij}) &= -\phi_{ij} y_{ij} = 0 && (12)\\
\phi_{ij} &\geq 0 && (13)
\end{aligned}
$$

From (10), (12), and (13), we obtain the condition below:

$$
\begin{cases} \eta_j - \eta_i = T_{ij}(y_{ij}, \tau_i) & (\text{if} \quad y_{ij} > 0) \\ \eta_j - \eta_i \leq T_{ij}(y_{ij}, \tau_i) & (\text{if} \quad y_{ij} = 0) \end{cases} \tag{14}
$$

Compared to (8), the dynamic equilibrium is established if η_i is equal to τ_i for all i.

As a whole, the dynamic problem forms a loop as illustrated below. Given $\boldsymbol{\tau}$, link flow $\mathbf{y}$ and the associated link travel time vector $\mathbf{T}$ are simultaneously obtained through optimization problem $P(\boldsymbol{\tau})$ as a function of $\boldsymbol{\tau}$. Then, from the link travel times, the earliest arrival times at nodes are obtained as the Lagrangean multiplier, $\boldsymbol{\eta}$, which must be the same as the assumed $\boldsymbol{\tau}$ for the equilibrium.

$$
\begin{tikzcd}
\boldsymbol{\eta} \arrow[d] & \mathbf{T} \arrow[l, dashed] \\
\boldsymbol{\tau} \arrow[r, dashed, "\mathrm{P}(\boldsymbol{\tau})"] & \mathbf{y} \arrow[u]
\end{tikzcd}
$$

As discussed so far, once arrival times τ_i's are known, the problem is reduced to the conventional static assignment. However, for a general network with queues, τ_i's cannot be known in advance except τ_r at origin r, which is clearly equal to s. Therefore, an iterative method should be normally applied to determine τ_i's as explained later.

There are however a few special cases in which arrival times τ_i's can be known without any difficulty. For instance, if none of the links has a queue during the concerned period of $[s - \delta s, s]$, the problem is apparently the same as the static assignment. An arrival time at a link can be immediately evaluated as the time later than the starting time s by the fastest free flow travel time to the link. Namely, the time period of $\tau_i(s) - \tau_i(s - \delta s)$ must be equal to δs for any node i, and equilibrium link flows y_{ij}'s are simply determined from $P(\boldsymbol{\tau})$.

For the other extreme, if queues form on all the links, the dynamic problem can be reduced to the simultaneous equations. Since the departure rate at every link must be its maximum rate with a queue, link flow y_{ij} is written as $\mu^*_{ij}(\tau_j(s) - \tau_j(s - \delta s))$; that is, L unknown y_{ij}'s can be described using (N-1) unknown arrival times of $\tau_j(s)$'s. Then, if this relationship is applied to flow conservation (7), (N-1) independent simultaneous equations with (N-1) unknown $\tau_j(s)$'s are obtained. Therefore, if queues were known to form on all the links, the dynamic problem could be completed by solving the simultaneous flow conservation equations.

5 Properties of the Dynamic Assignment

5.1 Existence of the Solution

The existence of the solution is proved based on the Brouwer's Fixed Point theorem. As explained in the previous section, the loop structure of the dynamic assignment is considered as the map of τ on itself through $P(\tau)$. For this type of problem, the Fixed Point theorem guarantees that if

1. the set of τ, S_τ, is closed convex and
2. the mapping from τ to τ is continuous,

then the fixed point exists.

Regarding the first condition, arrival time τ_i is basically evaluated from link travel times along the fastest route such that:

$$\tau_i = \min_k(\mathcal{T}_i^1, \mathcal{T}_i^2, \ldots.., \mathcal{T}_i^k, \ldots..., \mathcal{T}_i^p)$$

where

$$\begin{aligned} \mathcal{T}_i^k &= \text{arrival time to node } i \text{ via route } k \in S_k \\ &= \sum_{ij} T_{ij}\delta_{ij,k} + s, \qquad k = 1, 2, \ldots., p \\ \delta_{ij,k} &= \begin{cases} 1 & (\text{if route } k \text{ passes link (i,j)}) \\ 0 & (\text{otherwise}) \end{cases} \end{aligned}$$

T_{ij} clearly has lower and upper bounds, since T_{ij} must be larger than or equal to the free flow travel time m_{ij} and its maximum value must be finite because the traffic demand is finite. Then, a set of the arrival times via route k, $\mathcal{T}_i^k \in S_k$, is closed convex, since the T_{ij}'s are bounded and $\mathcal{T}_i^k$ is a linear combination of T_{ij}'s. Also, since τ_i is the minimum of $\mathcal{T}_i^k$, the set of τ_i denoted as S_τ is also closed convex (see Appendix 2).

For the second condition, we have to analyze optimization problem $P(\tau)$ to examine the continuity of the mapping. Based on the standard stability analysis on the optimization problem [19], if

(i) the constraints and the objective function of $P(\tau)$ are continuous in $\tau \in S_\tau$ and
(ii) a unique set of optimal link flow y_{ij} is obtained for any $\tau \in S_\tau$,

then the mapping is continuous in $\tau \in S_\tau$.

In our formulation, the objective function is indeed continuous in τ and the constraints do not contain τ. Thus, condition (i) is satisfied. Regarding condition (ii) on the uniqueness of the solution, it has been known that if the objective function is strictly convex in y_{ij}, a unique set of optimal link flows is obtained. However, our objective function is convex in link flow y_{ij} but not strictly convex, since link travel time T_{ij} is a constant value of m_{ij} without a queue on link (i,j). Thus, the solution of y_{ij}'s is not always unique and consequently the mapping would not be continuous in τ in general from the mathematical point of view.

As a result, the first condition on the convexity of a set of τ is satisfied, while the second condition on the continuity of the mapping is met provided that the objective function of $P(\tau)$ is strictly convex in y_{ij}. Then, the existence of the solution is guaranteed.

We fail to prove the existence property for the link travel time introduced here perhaps due to the approximation using the discretized starting times. However, it is difficult to find cases with no equilibrium solution from the engineering sense. An analysis from a different angle might provide a proof, which relies on the future research.

5.2 Sensitivity Analysis and the Uniqueness of the Solution

As discussed in the previous section, the whole problem has a loop structure. To find τ_i so as to be equal to η_i, we need know how η_i moves through $P(\tau)$, when input values τ_i's change.

Since the optimal solution must satisfy the Kuhn-Tucker condition, we can analyze the sensitivity of η_i through them. By differentiating (10), (11), and (12) with respect to τ_i's, derivative of $\nabla_\tau \eta$ is obtained as shown below (see Appendix 1 for the detailed derivation). Here, notation $\nabla_Q P$ generally stands for a $(m \times n)$ matrix representing the derivative of vector $P = (p_1, p_2, ..., p_m)$ with respect to vector $Q = (q_1, q_2, ..., q_n)$ in which $\partial p_i / \partial q_j$ is an (i,j) element of $\nabla_Q P$.

$$\nabla_\tau \eta \;=\; [\, \Omega^t \, J^{-1} \, \Omega \,]^{-1} \, \Omega^t J^{-1} \Theta \qquad (15)$$

where

$\nabla_\tau \eta$ = a $(N-1 \times N-1)$ matrix
J = a $(L \times L)$ Jacobian matrix of link travel time (diagonal),
Ω = a $(L \times N-1)$ link-node incidence matrix,
Θ = a $(L \times N-1)$ matrix whose row and column represent links and nodes. If the k th link is link (i,j), the (k,i) element of Θ is 1.

Because of the $N-1$ independent flow conservations as mentioned earlier, the symmetric matrix of $[\Omega^t J^{-1}\Omega]$ is $(N-1 \times N-1)$ and has its inverse because the rank of the link-node incidence matrix, Ω, is N-1 and the link time Jacobian, J, is diagonal.

As an example, let's consider a simple network with four nodes and five links shown in Fig.4, in which nodes 1 and 4 are the origin and destination nodes. The maximum departure rate of each link is shown left:

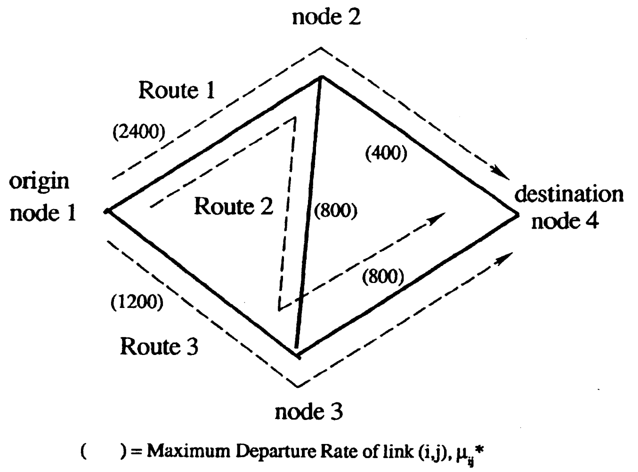

Fig.4 A Simple Network

Table 1. Maximum Departure Rate of Links

link sequential number	1	2	3	4	5
link (i,j)	(1,2)	(1,3)	(2,3)	(2,4)	(3,4)
μ^*_{ij} [veh/unit time]	2400	1200	800	400	800
m_{ij} [unit time]	1	1	1	1	1

In this example, Jacobian J is a (5×5) matrix, and Ω and Θ are (5×3) matrices as shown below. Originally, the incidence matrix is (5×4) matrix but here node 4 is eliminated because the rank is only 3. When all the links have queues, the inverse of Jacobian matrix J is written as

$$J^{-1} = \begin{pmatrix} 2400 & 0 & 0 & 0 & 0 \\ 0 & 1200 & 0 & 0 & 0 \\ 0 & 0 & 800 & 0 & 0 \\ 0 & 0 & 0 & 400 & 0 \\ 0 & 0 & 0 & 0 & 800 \end{pmatrix}$$

And matrices Ω and Θ are described below:

$$\Omega = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Theta = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, from (15), $\nabla_\tau \eta$ is written as:

$$\begin{aligned} \nabla_\tau \eta &= [\Omega^t J^{-1}\Omega]^{-1}\Omega^t J^{-1}\Theta \\ &= 1/400 \begin{pmatrix} 59/126 & 8/21 & 13/42 \\ 8/21 & 3/7 & 2/7 \\ 13/42 & 2/7 & 5/14 \end{pmatrix} \Omega^t J^{-1}\Theta \end{aligned}$$

To investigate the uniqueness of the solution, let $\psi(\tau)$ be the difference between η and τ, η - τ, which should be zero when the loop is closed. As seen from the schematic illustration of Fig.5, if $\psi(\tau)$ is monotone in τ, an intersection of $\psi(\tau)$ and the horizontal axis is unique. From (15), the derivative of $\psi(\tau)$ is given by:

$$\begin{aligned} \nabla_\tau \psi(\tau) &= \nabla_\tau \eta - I \\ &= [\, \Omega^t \, J^{-1} \, \Omega \,]^{-1} \, \Omega^t J^{-1} \Theta - I, \end{aligned} \tag{16}$$

where I is a $(L \times L)$ unit matrix (diagonal). If $\nabla_\tau \psi(\tau)$ is positive definite, the monotonicity is guaranteed. However, since the right hand side of this equation depends on Ω and Θ which represent the network configuration, it is difficult to generally prove the monotonicity of $\psi(\tau)$ and the uniqueness of the solution would not be warranted for a general network.

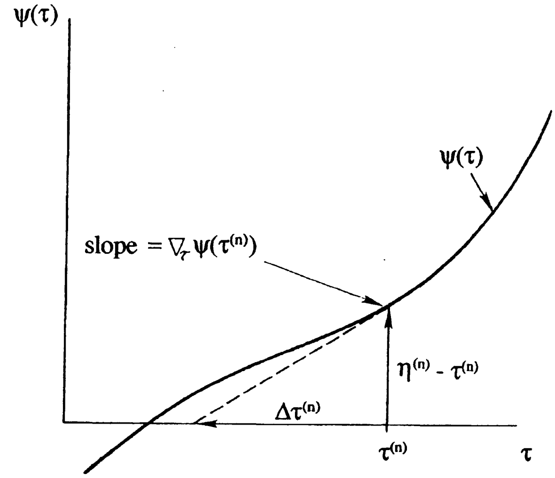

Fig.5 A Schematic Illustration of $\psi(\tau)$

5.3 Solution Algorithm

Suppose that, at the n th iteration, input vector $\tau^{(n)}$ yields $\eta^{(n)}$ through $P(\tau^{(n)})$. If the *Newton's* method is applied, the revised direction of vector $\tau^{(n)}$ denoted as $\Delta\tau^{(n)}$ is given by the condition:

$$\begin{gathered} \nabla_\tau \psi(\tau^{(n)}) \Delta\tau^{(n)} = -\psi(\tau^{(n)}) \\ [\, \nabla_\tau \eta^{(n)} - I \,] \Delta\tau^{(n)} = -[\, \eta^{(n)} - \tau^{(n)} \,] \end{gathered} \tag{17}$$

As you see from Fig.5, the slope $\nabla_\tau \psi(\tau^{(n)})$, which can be evaluated from the result of the sensitivity analysis (16), determines $\Delta\tau^{(n)}$ as an intersection of the slope and horizontal axis. The solution algorithm is briefly summarized below:

1. Let n be equal to 1 and assume initial vector $\tau^{(n)}$.
2. By solving optimization problem $P(\tau^{(n)})$, vectors $\mathbf{y}^{(n)}$ and $\eta^{(n)}$ are obtained.
3. If $\tau^{(n)}$ and $\eta^{(n)}$ is sufficiently close to each other, stop. Otherwise, calculate new vector $\tau^{(n+1)}$ based on (17) using appropriate step size α as shown below and return to step 2.

$$\tau^{(n+1)} = \tau^{(n)} + \alpha\Delta\tau^{(n)}, \quad 0 \leq \alpha \leq 1$$

For step 2, we can employ some efficient methods to solve the static assignment such as the $Frank - Wolfe$ algorithm to estimate $\mathbf{y}^{(n)}$ and $\eta^{(n)}$ without enumerating routes.

6 Examples

The proposed algorithm is applied to a simple network shown in Fig.4. The free flow travel time, m_{ij}, and the maximum departure rate, μ^*_{ij}, of each link are shown in Table 1. The OD demand from node 1 to 4 is assumed constant rate of 4800 [veh/ unit time] for all $t \geq 0$. We also assume that there is no vehicle on the network in the initial state at time 0 and the time interval of 0.1 [unit time] is employed for starting interval δs.

Fig.6 shows the convergence of route flows during time period of [0, 0.1] using the step size of $1/n$, where n is the iteration number. Initial values of τ_i's are set based on the free flow travel times. Basically, there are three alternative routes; however, route 2 via links (1,2), (2,3), and (3,4) takes 3 [unit time] without queues. Therefore, vehicles would choose either route 1 or 3 at first. In the equilibrium, the demand rate of 4800 [veh/ unit time] is divided into 1600 and 3200 on routes 1 and 3, and the cumulative arrival curves are drawn as shown in Fig.7, in which arrival times at node 1, 2, 3, and 4 are 0.1, 1.1, 1.27, 2.4 [unit time] respectively. In Fig.7, the cumulative arrivals and departures are classified by routes; that is, $A^k_{ij}(t)$ means the cumulative number of vehicles entering link (i,j) by time t via route k.

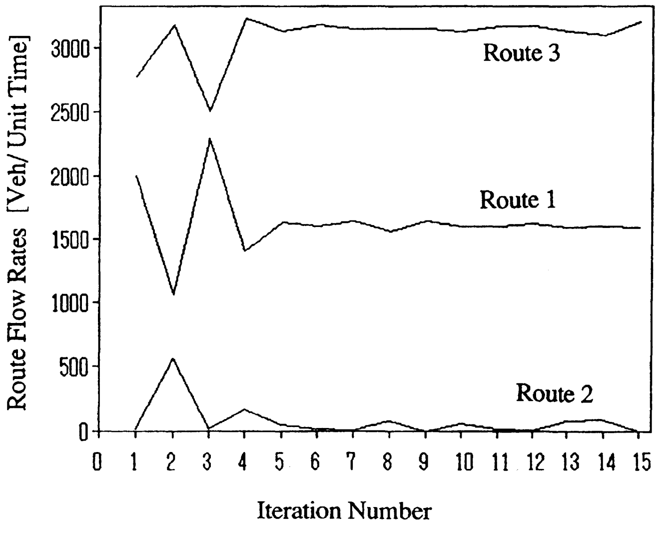

Fig.6 Convergence of the Route Flows

As a queue is growing on link (1,3) because of the demand rate of 3200 greater than the maximum departure rate of 1200 but not on link (1,2), the travel time to node 3 via link (1,3) becomes equal to one via links (1,2) and (2,3) at time 0.6 [unit time] as in Fig.7. That is, a vehicle starting the origin at time 0.6 arrives at node 3 at time 2.6 via both routes 2 and 3. Thus, after this time, all the three routes start having flows. The flow rate of three routes are then 1600, 1280, and 1920 [veh/ unit time].

Since all the links have queues after time 0.6 [unit time], the dynamic problem can be also considered as the simultaneous equation system as explained earlier. The flow conservation (7) at nodes 1, 2, and 3 are now written using the given maximum departure rates and unknown arrival times at the nodes. For instance, for the time period of [0.6, 0.7],

$$4800 \times 0.1 = 2400(\tau_2 - 1.6) + 1200(\tau_3 - 2.6),$$

$$2400(\tau_2 - 1.6) = 800(\tau_3 - 2.6) + 400(\tau_4 - 4.4),$$

$$800(\tau_3 - 2.6) + 1200(\tau_3 - 2.6) = 800(\tau_4 - 4.4).$$

Solving the simultaneous equations, we obtain the earliest arrival times at nodes for starting time s = 0.7 [unit time] (= τ_1) as below:

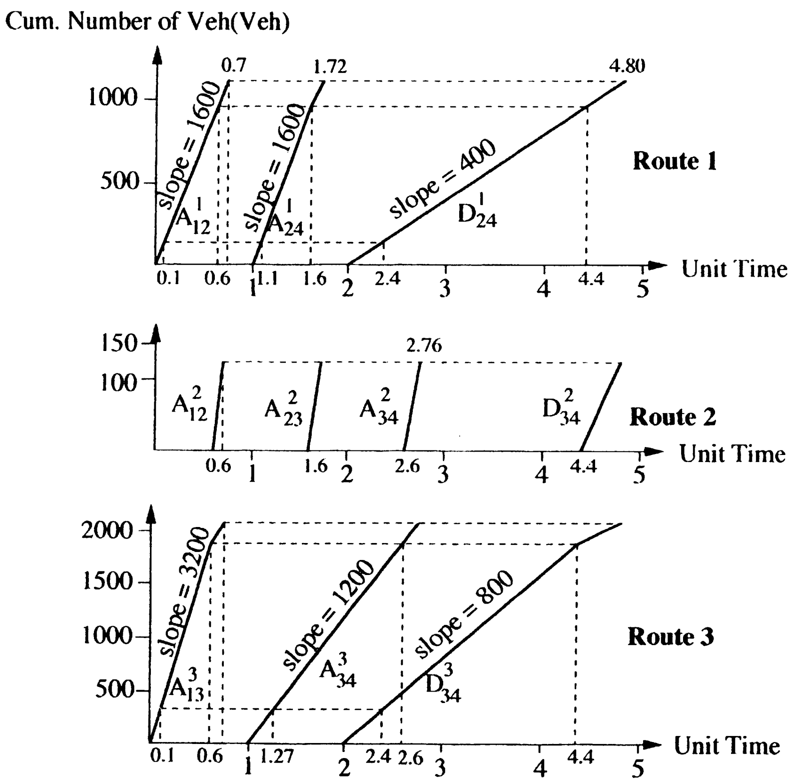

Fig.7 Cumulative Arrivals and Departures of the 3 Routes

(A^k_{ij} and D^k_{ij} mean the cum. arrivals and departures on link (i, j) via Route k)

$$\tau_2 = 1.72, \quad \tau_3 = 2.76, \quad and \quad \tau_4 = 4.80.$$

Although the solution is obtained in this case from simultaneous equations, we would not be generally able to know whether queues grow on all the links in advance for a more complex network.

7 Summary and Conclusion

This research deals with the dynamic traffic assignment on an over saturated network with queues for a one-to-many origin-destination pattern under the user equilibrium principle, given a time dependent OD demand. The major remarks are summarized below:

1. The dynamic assignment problem is decomposed with respect to the starting time from the single origin, since the order of vehicle arrivals at any node must be the same as the order in which those vehicles depart from the single origin, under the equilibrium. It is then shown that the problem forms the loop structure, which is formulated using two kinds of unknowns, link flows and arrival times at nodes, as an extension of the static assignment.

2. Based on the Fixed Point theorem, the solution is proved to exist provided that the link travel time is monotone increasing in link flow y_{ij}'s. We cannot success to prove the existence of the solution for the link travel time function used in this study, which takes a constant value of free flow travel time until a queue forms. On the other hand, the unique solution is not generally guaranteed in the dynamic equilibrium assignment.

3. A solution algorithm which does not require the route enumeration is proposed based on the sensitivity analysis of arrival times at nodes, and demonstrated in a simple network.

For future research, an analysis on the existence of the equilibrium solution from a different angle might relax the required condition concluded above so that the travel time function introduced here is warranted to have the solution. Also, the inclusion of the departure time choice in the dynamic equilibrium scheme in addition to the route choice, which may not be difficult, and the extension of the OD pattern to a many- to-many seem to be some of interesting future topics.

Appendix 1

Let F1, F2, and F3 be vectors whose elements are (10), (11), and (12) respectively as shown below:

$$\begin{aligned} F1 &= (\partial L/\partial y_{12}, ..., \partial L/\partial y_{ij}, ...), \\ F2 &= (h_1(\mathbf{y}), ..., h_j(\mathbf{y}), ...), \\ F3 &= (\phi_{12} g_{12}, ..., \phi_{ij} g_{ij}, ...). \end{aligned}$$

Derivatives of F1, F2, and F3 with respect to τ are written in a matrix form:

$$\begin{pmatrix} \nabla F1 \\ \nabla F2 \\ \nabla F3 \end{pmatrix} = \begin{pmatrix} \nabla_y F1 & \nabla_\eta F1 & \nabla_\Phi F1 \\ \nabla_y F2 & \nabla_\eta F2 & \nabla_\Phi F2 \\ \nabla_y F3 & \nabla_\eta F3 & \nabla_\Phi F3 \end{pmatrix} \begin{pmatrix} \nabla_\tau \mathbf{y} \\ \nabla_\tau \eta \\ \nabla_\tau \Phi \end{pmatrix} + \begin{pmatrix} \nabla_\tau F1 \\ \nabla_\tau F2 \\ \nabla_\tau F3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} J & \Omega & -I \\ \Omega^t & H & 0 \\ -\Phi & 0 & -Y \end{pmatrix} \begin{pmatrix} \nabla_\tau \mathbf{y} \\ \nabla_\tau \eta \\ \nabla_\tau \Phi \end{pmatrix} + \begin{pmatrix} -\Theta \\ 0 \\ 0 \end{pmatrix}$$

where

J = a $(L \times L)$ Jacobian matrix of link travel time (diagonal)
Ω = a $(L \times N-1)$ link-node incidence matrix
I = a $(L \times L)$ unit matrix (diagonal)
H = a $(N-1 \times N-1)$ diagonal matrix with elements of h_j's
Φ = a $(L \times L)$ diagonal matrix with elements of ϕ_{ij}'s
Y = a $(L \times L)$ diagonal matrix with elements of y_{ij}'s
Θ = a $(L \times N-1)$ matrix whose row and column represent links and nodes. If the k th link is link (i,j), the (k,i) element of Θ is 1.

Regarding (11), the number of independent constraints is N-1, only which we consider here.

The matrix description is rewritten as

$$\begin{aligned} J\nabla_\tau \mathbf{y} + \Omega \nabla_\tau \eta - I \nabla_\tau \Phi &= \Theta, & (18) \\ \Omega^t \nabla_\tau \mathbf{y} + H \nabla_\tau \eta &= 0, & (19) \\ -\Phi \nabla_\tau \mathbf{y} - Y \nabla_\tau \Phi &= 0. & (20) \end{aligned}$$

However, from the node conservation, $h_j(y)$ must be always zero which means H = 0. Also, if we consider only links with positive y_{ij}, ϕ_{ij} must be zero from the Kuhn-Tucker condition, and hence $\nabla_\tau \Phi = 0$. Then, (18) and (19) become

$$\nabla_\tau \mathbf{y} + J^{-1}\Omega\nabla_\tau \boldsymbol{\eta} = J^{-1}\Theta,$$

$$\Omega^t \nabla_\tau \mathbf{y} = 0.$$

Therefore,

$$\nabla_\tau \boldsymbol{\eta} \;=\; [\,\Omega^t\, J^{-1}\,\Omega\,]^{-1}\,\Omega^t J^{-1}\Theta.$$

Appendix 2

The earliest travel time to node i, $\tau_i \in S_\tau$ is given by

$$\tau_i = \min_k(\mathcal{T}_i^1, \mathcal{T}_i^2, \ldots.., \mathcal{T}_i^k, \ldots..., \mathcal{T}_i^p),$$

where a set of $\mathcal{T}_i^k$ denoted as S_k is closed convex. For $\tau_i \in S_\tau$ and $\tau_i' \in S_\tau$, if

$$\theta\tau_i + (1-\theta)\tau_i' \;\in S_\tau \quad\; 0 \le \theta \le 1,$$

a set of $\boldsymbol{\tau}$ can be said a closed convex set.

Suppose that τ_i and τ_i' are

$$\tau_i = \min_k(\mathcal{T}_i^1, \mathcal{T}_i^2, \ldots.., \mathcal{T}_i^k, \ldots..., \mathcal{T}_i^p),$$

$$\tau_i' = \min_k(\mathcal{T}_i^{1\prime}, \mathcal{T}_i^{2\prime}, \ldots.., \mathcal{T}_i^{k\prime}, \ldots..., \mathcal{T}_i^{p\prime}),$$

and both $\mathcal{T}_i^k \in S_k$ and $\mathcal{T}_i^{k\prime} \in S_k$. If $\mathcal{T}_i^m$ is the minimum of τ_i and τ_i':

$$\mathcal{T}_i^m = \min(\tau_i, \tau_i') = \min(\mathcal{T}_i^1, \ldots...., \mathcal{T}_i^p, \mathcal{T}_i^{1\prime}, \ldots...., \mathcal{T}_i^{p\prime}),$$

the following inequality is valid:

$$\begin{aligned}
\mathcal{T}_i^m &\le \theta\tau_i + (1-\theta)\tau_i' \\
&= \theta\mathcal{T}_i^m + (1-\theta)\tau_i' \\
&\le \tau_i' \\
&= \min_k(\mathcal{T}_i^{1\prime}, \mathcal{T}_i^{2\prime}, \ldots.., \mathcal{T}_i^{k\prime}, \ldots..., \mathcal{T}_i^{p\prime}) \\
&\le \mathcal{T}_i^{m\prime}.
\end{aligned}$$

Therefore,

$$\theta\tau_i + (1-\theta)\tau_i' \in S_m.$$

and consequently,

$$\begin{aligned}
\theta\tau_i \;&+\; (1-\theta)\tau_i' \\
&= \min(\mathcal{T}_i^{1\prime}, \mathcal{T}_i^{2\prime}, \ldots..., \mathcal{T}_i^{m-1\prime}, \theta\tau_i + (1-\theta)\tau_i', \mathcal{T}_i^{m+1\prime}, \ldots...., \mathcal{T}_i^{p\prime}).
\end{aligned}$$

Since all the elements of the right hand side are in S_k, $k = 1, 2, \ldots.., p$, $\theta\tau_i + (1-\theta)\tau_i'$ is in S_τ. Thus, S_τ is closed convex.

References

1. D.K. Merchant and G.L. Nemhauser : A Model and an Algorithm for the dynamic traffic assignment problem, Transportation Science, Vol.12, pp.183- 199, (1978).
2. D.K. Merchant and G.L. Nemhauser : Optimality Conditions for a Dynamic Traffic Assignment Model, Transportation Science, Vol.12, pp.200- 207, (1978).
3. J.H. Ho : A Successive Linear Optimization Approach to the Dynamic Traffic Assignment, Transportation Science, Vol.14, pp.295-305, (1980).
4. M. Carey : Optimal Time-Varying Flows on Congested Networks, Operations Research, Vol.35, pp.58-69, (1987).
5. T.L. Friesz, F.J. Luque, R.L. Tobin, and B.W. Wie : Dynamic NEtwork Traffic Assignment Considered as a Continuous Time Optimal Control Problem, Operations Research, VOl.37, pp.893-901, (1989).
6. B.W. Wie, T.L. Friesz, and R.L. Tobin : Dynamic User Optimal Traffic Assignment on Congested Multidestination Networks, Transportation Research, Vol.24, pp.431-442, (1990).
7. D.E. Boyce, B. Ran, and L.J. LeBlanc : Solving an Instantaneous Dynamic User Optimal Traffic Assignment Model, submitted to Transportation Science.
8. M.J. Smith and M.O. Ghali : Dynamic Traffic Assignment and Dynamic Traffic Control, Proceedings of the 11 th International Symposium on Transportation and Traffic Theory, Elsevier, pp.273-288, Yokohama, Japan, (1989).
9. M.J. Smith and M.O. Ghali : The Dynamics of Traffic Assignment and Traffic Control: a Theoretical Study, Transportation Research, Vol.24B, pp.409-422, (1990).
10. C.F.Daganzo, "The Uniqueness of a Time-Dependent Equilibrium Distribution of Arrivals at a Single Bottleneck", Transportation Science Vol.19, pp.29-37, (1985).
11. A.DePalma, M.Ben-Akiva, C.Lefevre, and N.Litinas, "Stochastic Equilibrium Model of Peak Period Traffic Congestion", Transportation Science Vol.17, pp.430-453, (1983).
12. C.Hendrickson and G.Kocur, "Schedule Delay and Departure Time Decisions in a Deterministic Model", Transportation Science Vol.15, pp.62- 77, (1981).
13. V.F.Hurdle, "Equilibrium Flows on Urban Freeways", Transportation Science Vol.5, pp.255-293, (1981).
14. M.Kuwahara, "A Time-Dependent Network Analysis for Highway Commute Traffic in a Single Core City", Ph.D thesis, University of California, Berkeley, (1985).
15. M.Kuwahara and G.F.Newell, "Queue Evolution on Freeways Leading to a Single Core City During the Morning Peak", in Proceedings of the Tenth International Symposium on Transportation and Traffic Theory, pp.21-40, Elsevier, (1987).
16. G.F.Newell, "The Morning Commute for Nonidentical Travelers", Transportation Science Vol.21, pp.74-88, (1987).
17. M.J.Smith, "The Existence of a Time-Dependent Equilibrium Distribution of Arrivals at a Single Bottleneck", Transportation Science Vol.18, pp.385-394, (1984).
18. D.J. Zawack and G.L. Thompson : A Dynamic Space-Tome Network Flow Models for City Traffic Congestion, Transportation Science, Vol.21, pp.153- 162, (1987).
19. M. Fukushima: Theory of Nonlinear Optimization Problems, Sangyo Tosho, Chap. 4, pp.83-102, (1980) (in Japanese).