TRANSPORTATION AND TRAFFIC THEORY

Proceedings of the 14th International Symposium on Transportation and Traffic Theory Jerusalem, Israel, 20-23 July, 1999

edited by

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PERGAMON

An Imprint of Elsevier Science
Amsterdam - Lausanne - New York - Oxford - Shannon - Singapore - Tokyo

A CAPACITY INCREASING PARADOX FOR A DYNAMIC TRAFFIC ASSIGNMENT WITH DEPARTURE TIME CHOICE

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ABSTRACT

This paper demonstrates that the capacity increasing paradox in a transportation networks as in Braess(1968) does also occur under non-stationary settings, in particular, under dynamic traffic assignment with endogenous time-varying origin-destination (OD) demands. Through the analyses, the analytical formulae for the solutions of the dynamic equilibrium assignment are explicitly derived for two kind of networks: the networks with a one-to-many OD pattern and the reversed networks with a many-to-one OD pattern; the formulae clarify the significant difference in the properties of the two dynamic flow patterns. This also leads us to the findings that one of the crucial conditions that determine whether the paradox occurs or not is the OD pattern of the underlying networks.

1. INTRODUCTION

Local improvements in a transportation network do not necessarily lead to the improvement of the global performance of the network. This fact has been well recognized as "Braess's paradox"(Braess (1968)) or "Smith's paradox"(Smith (1978)). The paradoxes stimulated many researchers in the field, and a considerable number of studies have been made on the relevant topics such as the network design problem or the sensitivity analysis of the equilibrium traffic assignment. Almost all the studies are, however, based on the framework of static (equilibrium) traffic assignment; only a few attempts have so far been made to study non-stationary (dynamic) traffic flow patterns with queues. Since the properties of the dynamic flow with queues are significantly different from those of the static flow without queues, many basic problems on the paradox under non-stationary settings are yet to be investigated.

The purpose of this paper is first to demonstrate that the capacity increasing paradox does also occur under non-stationary settings, in particular, under dynamic traffic assignment with endogenous, time-varying origin-destination (OD) demands. The paper also aims to capture the conditions that determine whether the paradox is likely to occur or not; we disclose that the OD pattern of the underlying networks is one of the crucial conditions.

In order to achieve the purpose, we first disclose that the *analytical* solution of the dynamic user equilibrium (we call this DUE) traffic assignment with elastic OD demands (i.e. the assignment considering users' departure-time choice behavior) can be obtained explicitly in a particular type of network satisfying some conditions. The solutions are derived for two kinds of network: (i) networks with single origin and multiple destinations (regarded as an "Evening rush hour" on a network of a city with a single CBD; we refer to this "E-net" hereafter); and (ii) networks with single destination and multiple origins (obtained by reversing the direction of all links and origin/destinations of the E-net, we may regard it as a "Morning rush hour" on the same network above; we refer to this "M-net"). Through the analyses of the two cases, we see the significant difference in the properties of the two dynamic flow patterns for not only the case where time-varying OD demands are given but also for the case of elastic OD demand due to user's departure time choice. These basic results for the DUE assignment then enables us to demonstrate the dynamic version of the capacity increasing paradox and to discuss the significant effect of OD pattern on the occurrence of the paradox.

The organization of this paper is as follows. In the second chapter, we briefly explain the basic properties of dynamic user equilibrium assignment, restricting ourselves to the minimum knowledge required for considering our problem. The third chapter explores the structure of the dynamic equilibrium assignment with exogenous OD demands for E-net and M-net. The analytical solution formulae of the equilibrium flow patterns for E-net and M-net are derived. The fourth chapter extends the analyses to the model with endogenous OD demand; not only the route choice but also the departure time choice are simultaneously considered in the model. For an appropriate set of boundary conditions, the explicit equilibrium flow patterns are derived for E-net and M-net. By using the results obtained in Chapters **3** and **4**, we demonstrate a dynamic version of Braess's paradox in the fifth chapter. We first discuss the paradox for the model with exogenous OD demand; the analysis on a simple network exhibits that the paradox arises *only on a* *particular condition* for the network with a one-to-many OD pattern, while the corresponding paradox *always* arises for the reversed network with a many-to-one OD pattern. We then show that the same results also hold for the model with endogenous OD demand. Finally, the last chapter summarizes the results and remarks on some further research topics.

2. DECOMPOSITION OF DYNAMIC EQUILIBRIUM ASSIGNMENT

2.1 Networks

Our model is defined on a transportation network $G[N, L, W]$ consisting of the set L of directed links with L elements, the set N of nodes with N elements, and the set W of origin-destination (OD) nodes pairs. The origins and the destinations are the subset of N, and we denote them by R and S, respectively. In this paper, we deal with only networks with a one-to-many OD (i.e. the element of R is unique) or those with a many-to-one OD (i.e. the element of S is unique). Sequential integer numbers from 1 to N are allocated to N nodes. A link from node i to j is denoted as link (i,j). We also use the notation to indicate a link by the sequential numbers from 1 to L allocated to all the links in the set L.

The structure of a network is represented by a node-link incidence matrix $\mathbf{A}^*$, which is an N $\times$L matrix whose (n, a) element is 1 if node n is an upstream-node of link a, -1 if node n is a downstream-node of link a, zero otherwise. The rank of this matrix is N-1 since the sum of rows in each column is always zero. Hence, it is convenient in representing our model to use the reduced incidence matrix $\mathbf{A}$ (instead of $\mathbf{A}^*$), which is an (N-1)$\times$L matrix eliminating an arbitrary row of $\mathbf{A}^*$. We call the node corresponding to the elimination "reference node". It is also convenient to "split" the matrix $\mathbf{A}$ into a pair of matrices, $\mathbf{A}_-$ and $\mathbf{A}_+$, defined as follows: $\mathbf{A}_-$ is a matrix that can be obtained by letting all the +1 elements of $\mathbf{A}$ be zero (i.e. the (n, a) element is -1 if link a arrives at node n, zero otherwise); $\mathbf{A}_+$ is a matrix that can be obtained by letting all the -1 elements of $\mathbf{A}$ be zero (i.e. the (n, a) element is +1 if link a leaves node n, zero otherwise); it is needless to say that $\mathbf{A} = \mathbf{A}_- + \mathbf{A}_+$ holds.

2.2. Link Model and Dynamic Equilibrium Assignment

For a link model in our dynamic assignment, we employ a First-In-First-Out (FIFO) principle and the point queue concept in which a vehicle has no physical length: it is assumed that the arrival flow at link (i,j) leaves the link after the free flow travel time m_{ij} if there exists no queue on the link, otherwise it leaves the link by the maximum departure rate (capacity) $\bar{\mu}_{ij}$.

Concerning the assignment principle, we assume the dynamic user equilibrium (DUE) assignment, which is a natural extension of the static user equilibrium assignment; the DUE is defined as the state where no user can reduce his/her travel time by changing his/her route unilaterally for an arbitrary time period.

2.3. Decomposition Property of Dynamic Equilibrium Assignment

Under the DUE state, the users who depart their origin at the same time, regardless of their routes, have the same arrival time at any node that is commonly passed through on the way to their destination. Furthermore, under the DUE state, the order of departure from the origin must be kept at any node through destinations. From these property, we can define the unique equilibrium arrival time at each node for each departure time from the origin.

As defined in the previous section, link travel time $c_{ij}(t)$ depends only on the vehicles which arrived at the link before time t. Therefore, together with the above discussion on the order of arrivals at a node, it is concluded that the travel time experienced by the vehicle that departs from an origin at time s is independent of the flows of the vehicles that depart from the origin after time s. Consequently, we can consider the assignment sequentially in the order of departure from the single origin. That is, the assignment can be decomposed with respect to the departure time from the single origin provided that the OD pattern is one-to-many. Similarly, for a many-to-one OD pattern, we can easily conclude that the assignment can be decomposed with respect to the arrival time at the single destination. For the detailed discussions on this property, see Kuwahara and Akamatsu (1993) and Akamatsu and Kuwahara (1994).

3. EQUILIBRIUM FLOW PATTERNS ON SATURATED NETWORKS - FIXED DEMAND CASE

In general, the DUE assignment is formulated as a non-linear complementarity problem (NCP) or a variational inequality problem (VIP), which implies that it is difficult to obtain the analytical properties of the assignment. Hence, instead of exploring the properties of the DUE assignment under general settings, we confine our analysis to "saturated networks" where we can obtain the *analytical solution*. The "saturated networks" are the networks satisfying the following two conditions: a) there exist inflows on all links over the network, b) there exist queues on all links over the network. The first condition a) is not very restrictive, since we can constitute the networks satisfying this condition after knowing the set of links with positive flows. Although the second condition b) may not be satisfied in many cases, we nevertheless employ this assumption because this assumption, as shown below, gives us the explicit formula for the solution of the DUE assignment, which enables us to understand the qualitative properties of interest.

We will first show the formulation for E-net and derive the solution in **3.1**; and then the formulation and the solution for M-net will be examined in **3.2**.

3.1. Equilibrium on Saturated Networks with a One-to-Many Pattern

(1) Formulation

The DUE assignment on a network with a one-to-many OD pattern can be decomposed with respect to the origin departure-time as mentioned in chapter **2**. Hence, once we know the method of solving the equilibrium pattern for one particular departure-time, we can obtain the equilibrium pattern for whole time periods by successively applying the same procedure at the order of the departure-time. In the following, we consider the problem of obtaining the equilibrium pattern for vehicles departing from origin *o* at time *s*, assuming that the solutions for vehicles departing before time *s* are already given.

In the decomposed formulation with origin departure time *s*, two kinds of variables, (y_{ij}^s, τ_i^s), play a central roll: τ_i^s is the earliest arrival time at node *i* for a vehicle departing from origin *o* at time *s*; y_{ij}^s is the link flow rate *with respect to s*, that is, $y_{ij}^s \equiv dF_{ij}(\tau_i^s)/ds$, where $F_{ij}(t)$ denote the cumulative number of vehicles entered into link *ij* at time *t*. In addition, we denote the number of vehicles with destination *d* departing from origin *o* until time *s* (cumulative OD demand by *departure-time*) by $Q_{od}(s)$.

In the DUE state, each user choose his/her route whose travel time is (*ex post*) minimum over the network. In other words, the links with positive inflows should be on the minimum path tree. In our saturated networks, all the links have positive inflows, and therefore the minimum path condition for users with origin departure-time *s* is written as $\mathbf{c}(s)+\mathbf{A}^T\boldsymbol{\tau}=\mathbf{0}$, where $\mathbf{c}(s)$ is an L dimensional column vector with elements $c_{ij}^s \equiv c_{ij}(\tau_i^s)$, $\boldsymbol{\tau}(s)$ is an (N-1) dimensional column vector with elements τ_i^s. Since the equation above should hold for any *s*, taking the derivative with respect to *s*, we have

$$\frac{d\mathbf{c}(s)}{ds}+\mathbf{A}^T\frac{d\boldsymbol{\tau}(s)}{ds}=0 \qquad \forall s, \tag{3.1}$$

where $d\mathbf{c}(s)/ds$ is an L dimensional column vector with elements dc_{ij}^s/ds, and $d\boldsymbol{\tau}(s)/ds$ is an N-1 dimensional column vector with elements $d\tau_i^s/ds$.

In our link model, the point queue and the FIFO principle are assumed, and therefore, the rate of change in link travel time is given by

$$\frac{dc_{ij}(t)}{dt}=\begin{cases}(\lambda_{ij}(t)/\bar{\mu}_{ij})-1 & \text{if there is a queue}\\ 0 & \text{otherwise}\end{cases}$$

where $\lambda_{ij}(t)$ is the standard link flow rate defined as $dF_{ij}(t)/dt$. Hence, in our saturated networks where all links have queues, the rate of change in the time needed to traverse link ij for users with origin departure time s, dc_{ij}^{s}/ds, can be represented as:

$$\frac{dc_{ij}^{s}}{ds}=\frac{dc_{ij}(\tau_{i}^{s})}{d\tau_{i}^{s}}\frac{d\tau_{i}^{s}}{ds}=\left(\frac{\lambda_{ij}(\tau_{i}^{s})}{\bar{\mu}_{ij}}-1\right)\frac{d\tau_{i}^{s}}{ds}.$$

Noticing here the definitional relationship $y_{ij}^{s}=\lambda_{ij}(\tau_{i}^{s})\cdot d\tau_{i}^{s}/ds$, we see that the dc_{ij}^{s}/ds reduces to a function of y_{ij}^{s} and τ_{i}^{s}:

$$\frac{dc_{ij}^{s}}{ds}=\frac{y_{ij}^{s}}{\bar{\mu}_{ij}}-\frac{d\tau_{i}^{s}}{ds}, \tag{3.2a}$$

or equivalently

$$\frac{d\mathbf{c}(s)}{ds}=\mathbf{M}^{-1}\,\mathbf{y}(s)-\mathbf{A}_{+}^{T}\frac{d\boldsymbol{\tau}(s)}{ds}\qquad\forall s. \tag{3.2b}$$

where $\mathbf{M}$ is a diagonal matrix whose ath diagonal element represents the maximum capacity of link a, $\mathbf{y}(s)$ is an L dimensional column vector with elements y_{ij}^{s}.

Substituting (3.2) into (3.1), we obtain

$$\mathbf{M}^{-1}\,\mathbf{y}(s)+(\mathbf{A}^{T}-\mathbf{A}_{+}^{T})\frac{d\boldsymbol{\tau}(s)}{ds}=\mathbf{0},\quad\forall s \tag{3.3}$$

and rearranging this yields

$$\mathbf{y}(s)=-\left(\mathbf{M}\mathbf{A}_{-}^{T}\right)\frac{d\boldsymbol{\tau}(s)}{ds}\qquad\forall s. \tag{3.4}$$

On the other hand, in the decomposed DUE formulation, the flow constraints that consist of the FIFO condition for each link and the flow conservation at each node over a network reduce to the following equations (for the detail, see Kuwahara and Akamatsu (1993), Akamatsu and Kuwahara (1994)):

$$-\mathbf{A}\,\mathbf{y}(s)-\frac{d\mathbf{Q}(s)}{ds}=\mathbf{0}\qquad\forall s. \tag{3.5}$$

where $d\mathbf{Q}(s)/ds$ is defined as an (N-1) dimensional vector with elements $dQ_{od}(s)/ds$ (given). Combining (3.5) with (3.4),

$$\left(\mathbf{A}\mathbf{M}\mathbf{A}_{-}^{T}\right)\frac{d\boldsymbol{\tau}(s)}{ds}=\frac{d\mathbf{Q}(s)}{ds}\qquad\forall s. \tag{3.6}$$

Thus, we see that the DUE assignment has a unique solution ($d\boldsymbol{\tau}(s)/ds$) if the rank of the matrix $\mathbf{A}\mathbf{M}\mathbf{A}_{-}^{T}$ is N-1.

(2) Solution

The rank of the matrix $\mathbf{A}\mathbf{M}\mathbf{A}_{-}^{T}$ generally depends on the choice of a reference node. For a network with a one-to-many OD, the rank of $\mathbf{AMA}_-^T$ can be less than N-1 when we choose an arbitrary node that is not an origin as the reference node. The rank, however, is always N-1 when an origin is employed as the reference node. Furthermore, since the value of $d\tau_i(s)/ds$ for an origin node is always 1 from the definition of $\tau_i(s)$, it is natural to choose an origin as the reference node. Thus, by setting an origin as the reference node, we obtain the equilibrium solution, $d\boldsymbol{\tau}(s)/ds$, by the following formula:

$$\frac{d\boldsymbol{\tau}(s)}{ds} = \left(\mathbf{AMA}_-^T\right)^{-1} \frac{d\mathbf{Q}(s)}{ds}. \tag{3.7}$$

In addition, we can obtain the equilibrium link flow pattern, $\mathbf{y}(s)$, by substituting (3.7) into (3.4).

3.2. Equilibrium on Saturated Networks with a Many-to-One Pattern

(1) Formulation

The DUE assignment on a network with a many-to-one OD pattern can be decomposed with respect to the destination arrival-time as shown in chapter **2**. In the following, we consider the problem of obtaining the equilibrium pattern for vehicles arriving at a destination at time *u*, assuming that the solutions for vehicles arriving before time *u* are already given.

For the networks with a many-to-one OD pattern, by decomposing with respect to the arrival time at a single destination, the discussions almost parallels to those in the previous section. In the decomposed formulation with destination arrival time *u*, two kinds of variables, (y_{ij}^u, τ_i^u), play a central roll: τ_i^u is the latest arrival time at node *i* for a vehicle reaching destination *d* at time *u*; y_{ij}^u is the link flow rate *with respect to u*, that is, $y_{ij}^u \equiv dF_{ij}(\tau_i^u)/du$. In addition, we denote the number of vehicles with origin *o* arriving at destination *d* until time *u* (cumulative OD demand by *arrival-time*) by $Q_{od}(u)$.

The formulation almost parallels the discussions in 3.1. First, the minimum path conditions for saturated networks reduces to the following conditions:

$$\frac{d\mathbf{c}(u)}{du} + \mathbf{A}^T \frac{d\boldsymbol{\tau}(u)}{du} = 0 \qquad \forall u. \tag{3.8}$$

Then the link travel time with a point queue for saturated networks also should satisfy

$$\frac{d\mathbf{c}(u)}{du} = \mathbf{M}^{-1}\,\mathbf{y}(u) - \mathbf{A}_+^T \frac{d\boldsymbol{\tau}(u)}{du} \qquad \forall u. \tag{3.9}$$

Substituting (3.9) into (3.8), we obtain

$$\mathbf{y}(u) = -\left(\mathbf{MA}_-^T\right)\frac{d\boldsymbol{\tau}(u)}{du} \qquad \forall u. \tag{3.10}$$

On the other hands , the link flow **y** should satisfy the flow constraints:

$$\mathbf{A}\,\mathbf{y}(u) - \frac{d\mathbf{Q}(u)}{du} = \mathbf{0} \qquad \forall u . \tag{3.11}$$

Combining (3.10) with (3.11), we reach

$$-\left(\mathbf{AMA}_{-}^{T}\right)\frac{d\boldsymbol{\tau}(u)}{du} = \frac{d\mathbf{Q}(u)}{du} \qquad \forall u . \tag{3.12}$$

Thus, we see that the DUE assignment has a unique solution ($d\boldsymbol{\tau}(u)/du$ and **y**(u)) if the rank of $\mathbf{AMA}_{-}^{T}$ is N-1.

(2) Solution

An arbitrary network with a many-to-one OD pattern can be obtained by reversing the direction of all links and origin/destinations of a network with a one-to-many OD pattern. Therefore, it is natural to expect that, "reversing" the result in 3.1, the rank of $\mathbf{AMA}_{-}^{T}$ become N-1 when a destination is chosen as the reference node. However, it is not the case for this problem; *the rank become less than N-1 even if we set the destination as the reference node; furthermore, we can prove that the rank is less than N-1 for any choice of the reference node.*

The reason why the rank of the matrix $\mathbf{AMA}_{-}^{T}$ becomes less than N-1 is that there exist particular origins (we call this "pure origins") that are not traversal nodes (i.e. the origin which has no links arriving at the origin). Letting B_{ij} be the (*i,j*) element of $\mathbf{A}^{*}\mathbf{MA}_{-}^{*T}$, we easily see that

$$B_{ij} = \begin{cases} -\bar{\mu}_{ij} & if \;\; i \neq j \\ \sum_{k} \bar{\mu}_{ki} & if \;\; i = j \end{cases} . \tag{3.13}$$

Hence, the column vectors of $\mathbf{AMA}_{-}^{T}$ corresponding to the pure origin are always zero, and the rank of $\mathbf{AMA}_{-}^{T}$ necessarily decreases by the number of pure origins.

To see this fact more precisely, we divide the node set N into two sub-sets: the set of pure origins, N_1, and the set of the other nodes, N_2. Then, we divide $\mathbf{A}^{*}$, $\mathbf{A}^{*}_{-}$, $d\tau(u)/du$ and $d\mathbf{Q}(u)/du$ into the two blocks corresponding to N_1 and N_2, respectively:

$$\mathbf{A}^{*} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}, \qquad \mathbf{A}_{-}^{*} = \begin{bmatrix} \mathbf{0} \\ \mathbf{A}_{2-} \end{bmatrix}, \quad \frac{d\boldsymbol{\tau}(u)}{du} = \begin{bmatrix} \dfrac{d\boldsymbol{\tau}_1(u)}{du} \\ \dfrac{d\boldsymbol{\tau}_2(u)}{du} \end{bmatrix}, \quad \frac{d\mathbf{Q}(u)}{du} = \begin{bmatrix} \dfrac{d\mathbf{Q}_1(u)}{du} \\ \dfrac{d\mathbf{Q}_2(u)}{du} \end{bmatrix},$$

where *i* th element of $d\mathbf{Q}_2(u)/du$ is defined as $-\sum_o \{dQ_{od}(u)/du\} = -\sum_k \mu_{kd}$ if *i* is an orign, $dQ_{id}(u)/du$ if *i* is a destination, zero otherwise. Note that $\mathbf{A}_{1\,-}$, which is the first block of $\mathbf{A}_{-}$ corresponding to N_1, is always **0** according to the definition of the pure origins. Rewriting (3.12) with these partitioned variables, we have

$$\begin{bmatrix} \dfrac{d\mathbf{Q}_1(u)}{du} \\ \dfrac{d\mathbf{Q}_2(u)}{du} \end{bmatrix} = \mathbf{A}^{\bullet}\mathbf{M}\mathbf{A}_{-}^{\bullet T} \frac{d\boldsymbol{\tau}(u)}{du} = \begin{bmatrix} \mathbf{0} & -\mathbf{A}_1\mathbf{M}\mathbf{A}_{2-}^{T} \\ \mathbf{0} & -\mathbf{A}_2\mathbf{M}\mathbf{A}_{2-}^{T} \end{bmatrix} \begin{bmatrix} \dfrac{d\boldsymbol{\tau}_1(u)}{du} \\ \dfrac{d\boldsymbol{\tau}_2(u)}{du} \end{bmatrix}. \tag{3.14}$$

That is,

$$\frac{d\mathbf{Q}_1(u)}{du} = -\mathbf{A}_1\mathbf{M}\mathbf{A}_{2-}^{T} \frac{d\boldsymbol{\tau}_2(u)}{du}, \tag{3.15a}$$

$$\frac{d\mathbf{Q}_2(u)}{du} = -\mathbf{A}_2\mathbf{M}\mathbf{A}_{2-}^{T} \frac{d\boldsymbol{\tau}_2(u)}{du}. \tag{3.15b}$$

This means that no condition which determines the $d\boldsymbol{\tau}_1 / du$ *for the pure origins is included in the equilibrium condition (3.12), while the* $d\boldsymbol{\tau}_2 / du$ *for the traversal nodes can be obtained by*

$$\frac{d\boldsymbol{\tau}_2(u)}{du} = -\left(\mathbf{A}_2\mathbf{M}\mathbf{A}_{2-}^{T}\right)^{-1} \frac{d\mathbf{Q}_2(u)}{du}. \tag{3.16}$$

Thus we see that *the solution of the DUE assignment with a many-to-one OD pattern can not be unique* and that for the problem to have a unique solution we should add appropriate conditions to resolve the indeterminacy of the $d\boldsymbol{\tau}_1 / du$.

4. EQUILIBRIUM FLOW PATTERNS ON SATURATED NETWORKS - ELASTIC DEMAND CASE

The previous chapter analyzed the solution of the DUE assignment where only user's route choice is endogenously described given time-varying OD demands. This chapter extends the analyses to the case where the time-dependent OD demands are endogenously determined (we call the model "DUE assignment with Elastic demand") by incorporating the user's departure time choice. The model employed here is the simplest one that consistently unifies the two kind of dynamic equilibrium models: the dynamic equilibrium assignment presented in the previous chapter and the dynamic equilibrium model of departure time choice as is well known since Vickrey (1969) or Hendrikson and Kocur (1980). For expositional brevity, the following assumptions are made in this paper:

1) The users with the same OD pair are homogeneous, that is, their utility functions are all the same and their desired arrival time is unique;
2) The users who arrive later than the desired arrival time do not exist [This is not a restrictive assumption but one just to make the exposition as simple as possible; it is easy to extend to the case where late arrival is permitted.].
3a) For the problems with one-to-many OD pattern (i.e. when we consider the problem on the basis of the origin departure-time), the disutility function for the users with destination d leaving origin at time s,

$V_{od}(s)$, is given as the linear combination of their travel time from the origin to destination d and their "schedule delay":

$$V_{od}(s) = a_1\{\tau_d(s) - s\} + a_2\{t_d - \tau_d(s)\}, \tag{4.1}$$

where a_1, a_2 are positive parameters that satisfy $a_1 > a_2$, $\tau_d(s)$ is the destination arrival-time for the users who start from origin at time s, and t_d is the users' desired arrival time.

3b) For the problems with many-to-one OD pattern (i.e. when we consider the problem on the basis of the destination arrival-time), the disutility function for the users with origin o arriving at the destination at time u, $V_{od}(u)$, is given as the linear combination of their travel time from origin o to the destination and their "schedule delay":

$$V_{od}(u) = a_1\{u - \tau_o(u)\} + a_2\{t_d - u\}, \tag{4.2}$$

where $\tau_o(u)$ is the origin departure-time for the users who arrive at destination at time u.

4) The networks can be regarded as "saturated networks" that is defined in the previous chapter.

4.1. Equilibrium on Saturated Networks with a One-to-Many Pattern

(1) Formulation

In this section we consider the networks with a one-to-many OD pattern where all nodes except the origin are destination, i.e., there are no nodes that are neither origin nor destination. [This is simply for the convenience of expositional brevity. The appropriate division of the node set easily extends our analyses to the general case where there are some nodes that are neither origin nor destination. See Appendix.]

The elastic demand DUE employed in this chapter is defined as the state where no one can improve his/her utility by changing either his/her route or their departure-time unilaterally. To formulate this, consider users who choose time s as departure time. Since the users choose their optimal route (conditional on the optimal departure time) in the DUE state, the equilibrium conditions for the route choice should be represented by the following differential equations as shown in Chapter **3**:

$$\left(\mathbf{AMA}_{-}^{T}\right)\frac{d\boldsymbol{\tau}(s)}{ds} = \frac{d\mathbf{Q}(s)}{ds}, \tag{4.3}$$

where the origin node is selected as a reference node as discussed in **3.1**. Then, the condition that no user can improve his utility by changing his/her departure-time in the DUE state can be represented by

$$\frac{\partial V_{od}(s)}{\partial s} = 0 \qquad \forall s,\ \forall d. \tag{4.4}$$

Substituting the definition of disutility function (4.1) into this, we obtain the equilibrium rate of change in the destination arrival-time as follows:

$$\frac{d\tau_d(s)}{ds} = \frac{a_1}{a_1 - a_2} \qquad \forall s,\ \forall d \tag{4.5}$$

[We are assuming that networks can be regarded as "saturated networks" and all OD pairs have positive OD flows during the period of time. In general we should consider the analysis period to include the time where some OD pairs have no generation of OD flows. By introducing appropriate classification, however, the general case can be reduced to the combination of our basic case (the case where all OD pairs have positive OD flows during the period for our analysis) and the case presented in Appendix.].

Thus, the elastic DUE conditions are represented as the following system of differential equations:

$$\begin{cases} \dfrac{d\boldsymbol{\tau}(s)}{ds} = \mathbf{E}\dfrac{a_1}{a_1 - a_2} & (4.6a) \\ \dfrac{d\mathbf{Q}(s)}{ds} = \left(\mathbf{AMA}_{-}^{T}\right)\dfrac{d\boldsymbol{\tau}(s)}{ds} & (4.6b) \end{cases}$$

where **E** is an (N-1) dimensional column vector whose elements are all equal to 1. It is worthwhile to compare the equilibrium conditions with those for the fixed demand case. In the fixed demand DUE model, eq.(4.3) with a given constant vector $d\mathbf{Q}(s)/ds$ determines $d\boldsymbol{\tau}(s)/ds$. On the contrast, in the elastic demand DUE, $d\boldsymbol{\tau}(s)/ds$ is first determined from the departure-time equilibrium condition, and then eq.(4.3) with fixed $d\boldsymbol{\tau}(s)/ds$ determines $d\mathbf{Q}(s)/ds$.

(2) Solution

By setting appropriate boundary conditions, we can obtain the solution $(\boldsymbol{\tau}(s), \mathbf{Q}(s))$ for the differential equation (4.6). For the boundary conditions, we first set the initial time $\hat{s}_s$ of the time period (measured with respect to the origin departure-time) during which eq.(4.6) holds (i.e. the networks can be regarded as "saturated networks" and all OD pairs have positive OD flows). Then we give the value of cumulative OD flows for the time $\hat{s}_s$ and for the final time of the period:

$$Q_{od}(\hat{s}_s) = \underline{Q}_{od} = given \qquad \forall d \tag{4.7a}$$

$$Q_{od}(s(t_d)) = \overline{Q}_{od} = given \qquad \forall d \tag{4.7b}$$

where $s(t_d)$ is an origin departure-time of the final users who arrive at destination d at time t_d (note that we do not have to give the value of $s(t_d)$ explicitly).

Integrating the second equation of (4.6) from time $\hat{s}_s$ to s with the initial condition (4.7a), we have

$$\mathbf{Q}(s) = \underline{\mathbf{Q}} + \left(\mathbf{AMA}_{-}^{T}\right)\mathbf{E}\frac{a_1}{a_1 - a_2}(s - \hat{s}_s), \tag{4.8}$$

where $\underline{\mathbf{Q}}$ is an (N-1) dimensional vector with elements $\underline{Q}_{od}$.

We then solve (4.6) with respect to $\boldsymbol{\tau}$. Integrating the first equation of (4.6) from time $\hat{s}_s$ to time $s(t_d)$ reduces to

$$\mathbf{t} - \boldsymbol{\tau}(\hat{s}_s) = \frac{a_1}{a_1 - a_2}\left(\mathbf{s}(t_d) - \mathbf{E}\,\hat{s}_s\right) \qquad \forall d. \tag{4.9}$$

where **t**, $\boldsymbol{\tau}(\hat{s}_s)$, and $\mathbf{s}(t_d)$ are (N-1) dimensional vectors with elements t_d, $\tau_d(\hat{s}_s)$, and $s(t_d)$, respectively.

The length of the time period that appears in the right hand side of (4.9), $s(t_d)-\hat{s}_s$, can be obtained by substituting (4.7b) into (4.8):

$$\mathbf{s}(t_d)-\mathbf{E}\,\hat{s}_s=\frac{a_1-a_2}{a_1}(\mathbf{AMA}_-^T)^{-1}(\overline{\mathbf{Q}}-\underline{\mathbf{Q}}). \tag{4.10}$$

Hence, from (4.10) and (4.9), we can determine the initial equilibrium arrival time corresponding to $\hat{s}_s$:

$$\boldsymbol{\tau}(\hat{s}_s)=\mathbf{t}-(\mathbf{AMA}_-^T)^{-1}(\overline{\mathbf{Q}}-\underline{\mathbf{Q}}) \tag{4.11}$$

Thus, the equilibrium pattern $(\boldsymbol{\tau}(s),\mathbf{Q}(s))$ with the boundary condition (4.7) is given by

$$\begin{cases}\boldsymbol{\tau}(s)=\left(\mathbf{t}-(\mathbf{AMA}_-^T)^{-1}(\overline{\mathbf{Q}}-\underline{\mathbf{Q}})\right)+\mathbf{E}\dfrac{a_1}{a_1-a_2}(s-\hat{s}_s) \\ \mathbf{Q}(s)=\underline{\mathbf{Q}}+\mathbf{AMA}_-^T\mathbf{E}\dfrac{a_1}{a_1-a_2}(s-\hat{s}_s)\end{cases}\quad \forall s, \tag{4.12}$$

and the corresponding equilibrium disutility $\tau_d(s)$ is calculated by

$$\boldsymbol{\rho}=(\mathbf{t}-\mathbf{E}\hat{s}_s)a_1+(\mathbf{AMA}_-^T)^{-1}(\overline{\mathbf{Q}}-\underline{\mathbf{Q}})(a_1-a_2)\qquad \forall s. \tag{4.13}$$

4.2. Equilibrium on Saturated Networks with a Many-to-One Pattern

(1) Formulation

In the following we consider the networks with a many-to-one OD pattern where all nodes except the destination are origins, i.e., there is no node that is neither origin nor destination. For the general case where there are some nodes that are neither origin nor destination, see Appendix.

We divide the node set N into two sub sets: the set of origins N_1, and the set of the single destination, N_2. Then, we divide $\mathbf{A}^*$, $\mathbf{A}^*_-$, $d\boldsymbol{\tau}(u)/du$ and $d\mathbf{Q}(u)/du$ into the two blocks corresponding to N_1 and N_2, respectively:

$$\mathbf{A}^*=\begin{bmatrix}\mathbf{A}_1\\ \mathbf{A}_2\end{bmatrix},\qquad \frac{d\boldsymbol{\tau}(u)}{du}=\begin{bmatrix}\dfrac{d\boldsymbol{\tau}_1(u)}{du}\\ 1\end{bmatrix},\qquad \frac{d\mathbf{Q}(u)}{du}=\begin{bmatrix}\dfrac{d\mathbf{Q}_1(u)}{du}\\ -\mu_d\end{bmatrix}. \tag{4.14}$$

where $\mathbf{A}_1$ is an (N−1)×L matrix, $\mathbf{A}_2$ is an L dimensional column vector, $d\mathbf{Q}_1(u)/du$ is an N-1 dimensional column vector with elements $dQ_{od}(u)/du$, and $\mu_d\equiv\sum_{ij\in L_d}\mu_{ij}$.

The elastic demand DUE employed here is defined as the state where no one can improve his/her utility by changing either his/her route or their departure/arrival-time unilaterally. Since the users choose their optimal route (conditional on having chosen his/her optimal departure/arrival-time) in the DUE state, the equilibrium conditions for the route choice should be represented by the following differential equations as shown in Chapter **3**:

$$\frac{d\mathbf{Q}(u)}{du} = -\left(\mathbf{A}^{\bullet}\mathbf{M}\mathbf{A}^{\bullet T}_{-}\right)\frac{d\boldsymbol{\tau}(u)}{du}. \tag{4.15}$$

Rewriting this with the variables introduced in (4.14), we have

$$\frac{d\mathbf{Q}_1(u)}{du} = -\left(\mathbf{A}_1\mathbf{M}\mathbf{A}^T_{1-}\right)\frac{d\boldsymbol{\tau}_1(u)}{du} - \left(\mathbf{A}_1\mathbf{M}\mathbf{A}^T_{2-}\right) \tag{4.16}$$

The condition that all the users can not improve their utility by changing his/her arrival-time (or departure-time) in the DUE state can be represented as

$$\frac{\partial V_{od}(u)}{\partial u} = 0 \qquad \forall u,\ \forall o. \tag{4.17}$$

Substituting the definition of disutility function (4.2) into this, we obtain the equilibrium rate of change in the destination arrival-time as follows: .

$$\frac{d\tau_o(u)}{du} = \frac{a_1 - a_2}{a_1} \qquad \forall u,\ \forall o \tag{4.18}$$

Thus, the elastic DUE conditions are represented as the following system of differential equations:

$$\begin{cases} \dfrac{d\boldsymbol{\tau}_1(u)}{du} = \mathbf{E}\dfrac{a_1 - a_2}{a_1} \\ \dfrac{d\mathbf{Q}_1(u)}{du} = -\left\{\left(\mathbf{A}_1\mathbf{M}\mathbf{A}^T_{1-}\right)\mathbf{E}\dfrac{a_1 - a_2}{a_1} - \left(\mathbf{A}_1\mathbf{M}\mathbf{A}^T_{2-}\right)\right\} \end{cases} \tag{4.19}$$

It is worthwhile to compare the equilibrium conditions with those for the fixed demand case. In the fixed demand DUE model, we tried to determine $d\boldsymbol{\tau}(u)/du$ from the eq.(4.15) with a given constant vector $d\mathbf{Q}(u)/du$. Then we encountered the indeterminacy of $d\boldsymbol{\tau}(u)/du$ due to the decrease in the rank of matrix $\mathbf{A}^{\bullet}\mathbf{M}\mathbf{A}^{\bullet}{-}$. On the contrast, in the elastic demand DUE, the indeterminacy problem is resolved since $d\boldsymbol{\tau}(u)/du$ is first determined from the departure-time equilibrium condition, and then eq.(4.16) with fixed $d\boldsymbol{\tau}(u)/du$ determines $d\mathbf{Q}(u)/du$.

(2) Solution

As in the case of one-to-many OD pattern, we can obtain the solution $(\boldsymbol{\tau}(s), \mathbf{Q}(s))$ for the differential equation (4.19) by giving appropriate boundary conditions. For the boundary conditions, we first set the initial time $\hat{u}_s$ of the time period (measured with respect to the destination arrival-time) during which eq.(4.19) holds (i.e. the networks can be regarded as "saturated networks" and all OD pairs have positive OD flows). Then, on a parallel with the discussion in 4.1, it is natural to give the value of cumulative OD flows from $\hat{u}_s$ and for the final time t_d :

$$Q_{od}(\hat{u}_s) = \underline{Q}_{od} = given \qquad \forall o, \tag{4.20a}$$

$$Q_{od}(t_d) = \overline{Q}_{od} = given \qquad \forall o, \tag{4.20b}$$

The conditions (4.20) in conjunction with (4.19) can be solved with respect to $\mathbf{Q}(u)$. However, these conditions are not enough to determine the value of $\boldsymbol{\tau}$. Hence, instead of (4.20a), we give the time needed to travel from origin o to the destination at the initial time $\hat{u}_s$ as a new boundary condition:

$$\hat{u}_s - \tau_o(\hat{u}_s) = r_{od} = given \qquad \forall o. \tag{4.20c}$$

Integrating the second equation of (4.19) from time u to t_d with the initial condition (4.20c), we have

$$\mathbf{Q}_1(u) = \overline{\mathbf{Q}} + \left\{ (\mathbf{A}_1\mathbf{M}\mathbf{A}_{1-}^T)\mathbf{E}\frac{a_1 - a_2}{a_1} + (\mathbf{A}_1\mathbf{M}\mathbf{A}_{2-}^T) \right\}(t_d - u) \quad \forall u \tag{4.21}$$

We next solve (4.19) with respect to $\boldsymbol{\tau}$. Integrating the first equation of (4.19) from time $\hat{u}_s$ to time u with the initial condition (4.20c), we obtain

$$\boldsymbol{\tau}(u) = (\mathbf{E}\,\hat{u}_s - \mathbf{r}) + \mathbf{E}\frac{a_1 - a_2}{a_1}(u - \hat{u}_s) \qquad \forall u. \tag{4.22}$$

and the corresponding equilibrium disutility $\tau_d(s)$ is calculated by

$$\boldsymbol{\rho} = a_2 \cdot (t_d - \hat{u}_s)\mathbf{E} + a_1\mathbf{r} \tag{4.23}$$

5. PARADOXES

Having derived the formulae for the solution of the dynamic traffic equilibrium assignment so far, now we can discuss the capacity increasing paradox. The paradox presented here is a situation such that improving the capacity of a certain link on a network worsen the total travel cost over the network; this is a dynamic version of Braess's paradox which is well known in the static assignment. Using the results obtained in Chapters 3 and 4, we derive the necessary conditions for the occurrence of the paradox for E-net and M-net, which are shown to be significantly different.

5.1. A Paradox for a Network with a One-to-Many OD Pattern

We consider the paradox for the network shown in Fig. 5.1, where node 1 is a unique origin; nodes 2 and 3 are destinations; the maximum departure rate of link a (a = 1,2,3) is given by μ_a.

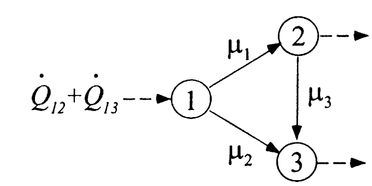

Fig.5.1. Example Network with Single Origin and Two Destinations

For the brevity of notation, we employ the superscript " • " as the derivative operation with respect to origin departure-time s in this section. (e.g. $\dot{\tau}_i(s) \equiv d\tau_i(s)/ds$, $\dot{Q}_{od}(s) \equiv dQ_{od}(s)/ds$).

(1) Fixed Demand Case

For the network in Fig. 5.1, the origin (i.e. node 1) should be the reference node; the incidence matrix $\mathbf{A}^*$, the reduced incidence matrix $\mathbf{A}$, and the corresponding $\mathbf{A}\_$ are given as follows:

$$\mathbf{A}^* = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}, \quad \mathbf{A}_- = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \tag{5.1}$$

Hence,

$$\mathbf{AMA}_-^T = \begin{bmatrix} \mu_1 & -\mu_3 \\ 0 & \mu_2 + \mu_3 \end{bmatrix}, \quad \left(\mathbf{AMA}_-^T\right)^{-1} = \begin{bmatrix} \dfrac{1}{\mu_1} & \dfrac{\mu_3}{\mu_1(\mu_2+\mu_3)} \\ 0 & \dfrac{1}{\mu_2+\mu_3} \end{bmatrix}. \tag{5.2}$$

The equilibrium pattern for the vehicles with the departure time s from a single origin can be calculated using the results of Chapter 3. From (3.6), we first obtain the rate of change in equilibrium arrival time:

$$\dot{\tau}_2(s) = \frac{1}{\mu_1}\dot{Q}_{12}(s) + \frac{\mu_3}{\mu_1(\mu_2+\mu_3)}\dot{Q}_{13}(s), \quad \dot{\tau}_3(s) = \frac{1}{\mu_2+\mu_3}\dot{Q}_{13}(s) \tag{5.3}$$

Substituting these into (3.3), we have the following equilibrium link flow pattern:

$$y_1(s) = \dot{Q}_{12}(s) + \frac{\mu_3}{\mu_2+\mu_3}\dot{Q}_{13}(s) = \dot{Q}_{12}(s) + y_3(s)$$

$$y_2(s) = \frac{\mu_2}{\mu_2+\mu_3}\dot{Q}_{13}(s), \quad y_3(s) = \frac{\mu_3}{\mu_2+\mu_3}\dot{Q}_{13}(s). \tag{5.4}$$

To discuss the "capacity increasing paradox", we employ the total travel time for the users departing from an origin from time *0* to T as an indicator for measuring the efficiency of the network flow pattern:

$$TC \equiv \sum_a \int_0^T y_a(s)c_a(s)ds = \sum_d \int_0^T \dot{Q}_{od}(s)\{\tau_d(s) - s\}ds \tag{5.5}$$

We then refer to the situation "paradox" if increasing the capacity of a certain link, μ_a, causes the increase of TC (i.e. $dTC/d\mu_a > 0$ implies "paradox").

Let us examine whether the paradox arises or not for the network in Fig. 4.1. Substituting (5.3) into (5.5), we obtain TC :

$$TC = \int_0^T \left[\dot{Q}_{12}(s)\left\{ \frac{Q_{12}(s)}{\mu_1} + \frac{\mu_3 Q_{13}(s)}{\mu_1(\mu_2+\mu_3)} + \tau_2(0) - s \right\} + \dot{Q}_{13}(s)\left\{ \frac{Q_{13}(s)}{\mu_2+\mu_3} + \tau_3(0) - s \right\} \right] ds \quad (5.6)$$

From (5.6), we easily see that the increase of μ_1 or μ_2 always decreases TC (note that both μ_1 and μ_2 appear in only the denominator of TC), that is, the paradox does not arise for links 1 and 2. Increasing μ_3, however, causes the paradox. The reason is that since

$$\frac{dTC}{d\mu_3} = \left[\mu_2 \left\{ \int_0^T \dot{Q}_{12}(s) Q_{13}(s) ds \right\} - \mu_1 \left\{ \int_0^T \dot{Q}_{13}(s) Q_{13}(s) ds \right\} \right] \frac{1}{\mu_1(\mu_2+\mu_3)^2}, \quad (5.7)$$

if the condition:

$$\frac{\int_0^T \dot{Q}_{12}(s) Q_{13}(s) ds}{\mu_1} > \frac{\int_0^T \dot{Q}_{13}(s) Q_{13}(s) ds}{\mu_2} \quad (5.8)$$

holds, $dTC/d\mu_3$ *is always positive, this means the occurrence of the paradox.*

The (5.8) is the condition that the paradox occurs for a certain time period *0* ~ *T.* From this, we can also derive the condition under which the paradox occurs for an arbitrary time period:

$$\dot{Q}_{12}(s)/\mu_1 > \dot{Q}_{13}(s)/\mu_2 . \quad (5.9)$$

The meaning of this inequality is simple. Since the increase of μ_3 always results in the increase of y_3 (see (5.4)), suppose 1 unit of increase in flow on link 3 (= y_3). This means that the number of users with destination 3 who pass through link 1 increases by 1 unit. The increase in flow on link 1 then causes $\dot{Q}_{12}(s)/\mu_1$ of increases in total travel time for the users with destination 2 ("User-2"). On the other hand, total travel time for the users with destination 3 ("User-3") decreases by $\dot{Q}_{13}(s)/\mu_2$, since the flow on link 2 decreases 1 unit. Therefore, the 1 unit of increase in flow on link 3 causes the increase of total travel time by $\dot{Q}_{12}(s)/\mu_1 - \dot{Q}_{13}(s)/\mu_2$. Thus, we see that (5.9) means the condition that the "net benefit" for User-2 and User-3 (User-3's benefit minus User-2's loss) due to the increase of μ_3 becomes positive.

(2) Elastic Demand Case

The equilibrium pattern for the network in Fig. 5.1 can be calculated from the results of Chapter 4. From (4.12), we first obtain the equilibrium arrival times and OD flows:

$$\tau_3(s) = \frac{a_1}{a_1 - a_2}(s - \hat{s}_s) + \left(t_3 - \frac{1}{\mu_2+\mu_3}\hat{Q}_{13} \right) \quad (5.10)$$

$$Q_{12}(s) = Q_{12}(\hat{s}_s) + (\mu_1 - \mu_3)\frac{a_1}{a_1-a_2}(s - \hat{s}_s), \; Q_{13}(s) = Q_{13}(\hat{s}_s) + \mu_2 \frac{a_1}{a_1-a_2}(s - \hat{s}_s). \quad (5.11)$$

where $\hat{Q}_{od} \equiv \overline{Q}_{od} - \underline{Q}_{od}$. Then (4.13) gives the equilibrium disutility for each origin:

$$\rho_2 = a_1(t_2 - \hat{s}_s) + (a_1 - a_2)\frac{1}{\mu_1 - \mu_3}\hat{Q}_{12}, \quad \rho_3 = a_1(t_3 - \hat{s}_s) + (a_1 - a_2)\frac{1}{\mu_2 + \mu_3}\hat{Q}_{13}. \qquad (5.12)$$

We define the sum of disutility experienced by all users over a network, *TC*, as an indicator for measuring the efficiency of the network usage:

$$TC \equiv \sum_d \rho_d \hat{Q}_{od}. \qquad (5.13)$$

The *TC* for the network in Fig.5.1 is given by

$$TC = a_1\{(t_2 - \hat{s}_s)\hat{Q}_{12} + (t_3 - \hat{s}_s)\hat{Q}_{13}\} + (a_1 - a_2)\left(\frac{\hat{Q}_{12}^2}{\mu_1 - \mu_3} + \frac{\hat{Q}_{13}^2}{\mu_2 + \mu_3}\right) \qquad (5.14)$$

To check the occurrence of the paradox, we calculate $dTC/d\mu_3$:

$$\frac{dTC}{d\mu_3} = \left(\frac{\hat{Q}_{12}}{\mu_1 - \mu_3}\right)^2 - \left(\frac{\hat{Q}_{13}}{\mu_2 + \mu_3}\right)^2 \qquad (5.15)$$

Note that the capacity of link 1 should be greater than that of link 3 (i.e. $\mu_1 > \mu_3$) in order for (5.11) to satisfy the (physically evident) condition $Q_{12}(s) - Q_{12}(\hat{s}_s) > 0$. Hence $dTC/d\mu_3 > 0$ holds only if

$$\hat{Q}_{12}/(\mu_1 - \mu_3) > \hat{Q}_{13}/(\mu_2 + \mu_3). \qquad (5.16a)$$

We see from (5.16a) that the paradox arise (with the capacity increase of link 3) independent of the value of μ_3 if the following condition hold:

$$\hat{Q}_{12}/\mu_1 > \hat{Q}_{13}/\mu_2. \qquad (5.16b)$$

It is noteworthy that the condition (5.16b) is identical in form to the condition for the fixed demand case.

5.2. A Paradox for a Network with a Many-to-One OD Pattern

We consider the paradox for the network in Fig.5.2, where node 1 is a unique destination; nodes 2 and 3 are origins; the maximum departure rate of link a (a = 1,2,3) is given by μ_a.
For the brevity of notation, we employ the superscript " • " as the derivative operation with respect to destination arrival time u in this section. (e.g. $\dot{\tau}_i(u) \equiv d\tau_i(u)/du$, $\dot{Q}_{od}(u) \equiv dQ_{od}(u)/du$)

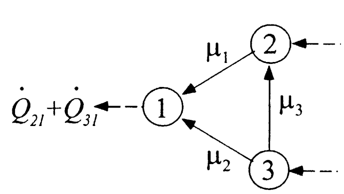

Fig. 5.2. Example Network with Two Origins and Single Destination

(1) Fixed Demand Case

For the network in Fig. 5.2, node 3 is the pure origin; we divide the incidence matrix $\mathbf{A}^{\bullet}$, the corresponding $\mathbf{A}^{\bullet}_{-}$ and the OD flow vector as follows:

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} (node\,3) \quad \mathbf{A}_{1-} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad d\mathbf{Q}_1(u)/du = [\dot{Q}_{31}(u)]$$
$$\mathbf{A}_2 = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{matrix} (node\,1) \\ (node\,2) \end{matrix}, \quad \mathbf{A}_{2-} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad d\mathbf{Q}_2(u)/du = \begin{bmatrix} -(\mu_1+\mu_2) \\ \dot{Q}_{21}(u) \end{bmatrix}. \tag{5.17}$$

Hence,

$$\mathbf{A}^{\bullet}\mathbf{M}\mathbf{A}^{\bullet T}_{-} = \begin{bmatrix} 0 & \mathbf{A}_1\mathbf{M}\mathbf{A}_{2-}^T \\ \mathbf{0} & \mathbf{A}_2\mathbf{M}\mathbf{A}_{2-}^T \end{bmatrix} = \left[\begin{array}{c|cc} 0 & -\mu_2 & -\mu_3 \\ \hline 0 & \mu_1+\mu_2 & 0 \\ 0 & -\mu_1 & \mu_3 \end{array}\right]. \tag{5.18}$$

The equilibrium pattern for the vehicles with the arrival time u at a single destination can be calculated from the results in Chapter 3. From (3.16), we first obtain the rate of change in equilibrium arrival time for nodes 1 and 2:

$$\dot{\tau}_1(u) = \frac{\mu_1+\mu_2}{\mu_1+\mu_2} = 1, \qquad \dot{\tau}_2(u) = \frac{\mu_1 - \dot{Q}_{21}(u)}{\mu_3}. \tag{5.19}$$

Substituting these into (3.10) yields the link flow rates (with respect to u):

$$y_1(u) = \mu_1, \quad y_2(u) = \mu_2, \quad y_3(u) = \mu_1 - \dot{Q}_{21}(u) \tag{5.20}$$

Note that this flow pattern is significantly different from that for the reversed network (see (5.4)).

In order to determine the rate of change in equilibrium arrival time for node 3 (= the pure origin), adding an appropriate condition is required. Here we assume for node 3 that the OD flow rate *measured at the origin*, $\hat{q}_{31} \equiv dQ_{31}(u)/d\tau_3(u) = \dot{Q}_{31}(u)/\dot{\tau}_3(u)$, is given. On the other hand, the OD flow rate *measured at the destination*, $q_{31} \equiv \dot{Q}_{31}(u)$, is determined from (3.15a):

$$\dot{Q}_{31}(u) = \mu_1 + \mu_2 - \dot{Q}_{21}(u) \tag{5.21}$$

Substituting this into the definitional relationship between $\hat{q}_{od}$ and q_{od}:

$$\frac{q_{od}(u)}{\hat{q}_{od}(u)} = \frac{dQ_{od}(u)}{du} \Big/ \frac{dQ_{od}(u)}{d\tau_o(u)} = \frac{d\tau_o(u)}{du},$$

we obtain the rate of change in equilibrium arrival time at node 3:

$$\dot{\tau}_3(u) = \frac{\dot{Q}_{31}(u)}{\hat{q}_{31}(u)} = \frac{\mu_1 + \mu_2 - \dot{Q}_{21}(u)}{\hat{q}_{31}(u)}. \tag{5.22}$$

Defining the total travel time for the users arriving at an destination from time *0* to *T* as an indicator for measuring the efficiency of the network flow pattern:

$$TC \equiv \sum_a \int_0^T y_a(u)c_a(u)du = \sum_o \int_0^T \dot{Q}_{od}(s)\{u - \tau_o(u)\}du, \tag{5.23}$$

let us examine whether the paradox arises or not in the network in Fig. 5.2. Substituting (5.19), (5.21) and (5.22) into (5.23), we obtain the *TC* for this network:

$$TC = \int_0^T \left[\dot{Q}_{21}(u)\left\{ u - \frac{\mu_1 u - Q_{21}(u)}{\mu_3} - \tau_2(0) \right\} + \dot{Q}_{31}(u)\left\{ u - \frac{(\mu_1+\mu_2)u - Q_{21}(u)}{\hat{Q}_{31}} - \tau_3(0) \right\} \right] du \tag{5.24}$$

where $\hat{Q}_{31}(u) \equiv \int_0^T \hat{q}_{od}(u)du$. We see from this equation that the increase in μ_1 or μ_2 will always decrease *TC*; the paradox does not arise for links 1 and 2. However, the increase in the capacity of link 3 *always* results in the occurrence of the paradox. This fact can be easily examined as follows. Calculating the derivative of TC with respect to μ_3, we have

$$\frac{dTC}{d\mu_3} = \int_0^T \dot{Q}_{21}(u)\frac{\mu_1 u - Q_{21}(u)}{\mu_3^2}du = \frac{1}{\mu_3^2}\int_0^T \left\{ \dot{Q}_{21}(u)\int_0^u \dot{\tau}_2(t)dt \right\} du. \tag{5.25}$$

Note that $\dot{\tau}_2(u)$ should be positive in the DUE state. The reason is that if $\dot{\tau}_2(u)$ is not positive the users with the destination arrival time *u'*> *u* must depart from their origin before the users with arrival time *u*, and this contradict the assumption that the state is in the DUE. Therefore, from the (5.25) and the fact that $\dot{\tau}_2(u)>0$ for any *u*, the inequality $dTC/d\mu_3 > 0$ *always* holds; we see that the paradox for link 3 takes place *without any additional conditions*.

(2) Elastic Demand Case

The equilibrium pattern for the network in Fig. 5.2 can be calculated from the results of Chapter 4. For the network in Fig.5.2, the matrices $\mathbf{A}_1\mathbf{M}\mathbf{A}_{1-}^T$ and $\mathbf{A}_2\mathbf{M}\mathbf{A}_{2-}^T$ defined in **4.2** are

$$\mathbf{A}_1\mathbf{M}\mathbf{A}_{1-}^T = \begin{bmatrix} \mu_3 & 0 \\ -\mu_3 & 0 \end{bmatrix}, \qquad \mathbf{A}_1\mathbf{M}\mathbf{A}_{2-}^T = \begin{bmatrix} -\mu_1 \\ -\mu_2 \end{bmatrix}. \tag{5.26}$$

Hence, from (4.21) and (4.22), we obtain the equilibrium arrival times and OD flows:

$$\tau_2(u) = \frac{a_1 - a_2}{a_1}u + \frac{a_2}{a_1}\left\{ \hat{u}_s - \frac{a_1}{a_2}r_{21}(\hat{u}_s) \right\}, \quad \tau_3(u) = \frac{a_1 - a_2}{a_1}u + \frac{a_2}{a_1}\left\{ \hat{u}_s - \frac{a_1}{a_2}r_{31}(\hat{u}_s) \right\} \tag{5.27}$$

$$Q_{21}(u) = \overline{Q}_{21} + \frac{(a_1 - a_2)\mu_3 - a_1\mu_1}{a_1}(t-u), \quad Q_{31}(u) = \overline{Q}_{31} - \frac{(a_1 - a_2)\mu_3 + a_1\mu_2}{a_1}(t-u) \tag{5.28}$$

We also get the equilibrium disutility from (4.23):

$$\rho_2 = a_2(t - \hat{u}_s) + a_1 r_{21}(\hat{u}_s), \quad \rho_3 = a_2(t - \hat{u}_s) + a_1 r_{31}(\hat{u}_s). \tag{5.29}$$

Let us define the sum of disutility experienced by all users over a network, *TC*, as an indicator for measuring the efficiency of the network usage:

$$TC \equiv \sum_{o} \rho_o \hat{Q}_{od} \tag{5.30}$$

Substituting (5.29) into the definition (5.30), we get the TC for the network in Fig.5.2:

$$\begin{aligned} TC &= \rho_2\left(\overline{Q}_{21} - Q_{21}(\hat{u}_s)\right) + \rho_3\left(\overline{Q}_{31} - Q_{31}(\hat{u}_s)\right) \\ &= -\left\{a_2(t-\hat{u}_s) + a_1 r_{21}(\hat{u}_s)\right\}\left\{\frac{(a_1 - a_2)\mu_3 - a_1\mu_1}{a_1}(t-\hat{u}_s)\right\} \\ &\quad + \left\{a_2(t-\hat{u}_s) + a_1 r_{31}(\hat{u}_s)\right\}\left\{\frac{(a_1 - a_2)\mu_3 + a_1\mu_2}{a_1}(t-\hat{u}_s)\right\} \end{aligned} \tag{5.31}$$

To check the occurrence of the paradox, we calculate $dTC/d\mu_3$:

$$\frac{dTC}{d\mu_3} = a_1(a_1 - a_2)(t - \hat{u}_s)\{r_{31}(\hat{u}_s) - r_{21}(\hat{u}_s)\}. \tag{5.32}$$

Note that the relationship

$$r_{31}(\hat{u}_s) > r_{21}(\hat{u}_s) \tag{5.33}$$

or equivalently,

$$\tau_2(\hat{u}_s) > \tau_3(\hat{u}_s) \tag{5.34}$$

should holds as long as the network in Fig.5.2 is a saturated network. The reason can be proved by contradiction: consider two users with origin 2 and 3, denoted as U2 and U3, who arrive at the destination at the same time $\hat{u}_s$; suppose that the (5.34) does not hold, then it implies that U2 should leave his origin earlier than U3 does; this clearly contradict the assumption of the saturated network. Thus, from (5.32) and (5.33), we see that $dTC/d\mu_3 > 0$ *always* holds; in other words, the occurrence of the paradox is inevitable when the capacity of link 3 is increased. It is worth noting that we eventually obtained the same result as in the fixed demand case.

6. Concluding Remarks

This paper discussed a capacity increasing paradox under a dynamic equilibrium assignment with elastic OD demands: the paradox is a situation such that improving the capacity of a certain link on a network worsen the total travel cost over the network. Our analysis in a simple network disclosed that the paradox arises *only on a particular condition* for a network with a one-to-many OD pattern, while the corresponding paradox *always* arises for the reversed network with a many-to-one OD pattern. This is the asymmetrical result that can not be seen in the classical static assignment framework; it is particular to the dynamic assignment with queue. Furthermore, we show that this property holds not only for the assignment with fixed OD demands but also for the assignment with elastic OD demands.

In this paper, particular simple networks were employed to demonstrate the paradox. Note, however, that the examples presented here are not the exceptional ones that can hardly be observed in practical situations but the ones that can be seen universally if we regard the example networks as a macroscopic representation of real road networks. Therefore, we think that the examples, despite their simplicity, describe one of the essential points that should be considered in deciding practical traffic management operations such as ramp metering or addition of lanes in freeways.

We recognize that there are still several relevant topics to be studied. First, we should extend our analysis to the paradox in a more complex network by exploiting the analytical formula of the DUE solution derived in this paper; it may be possible to obtain systematic methods for general networks that detect (without computing the equilibrium patterns) the links where the paradox takes place; the exploration of this possibility would be an interesting future topic. Secondly, we should analyze more realistic case where the assumption of "saturated networks" are relaxed; the exploration would be achieved by employing not only the analytical approach just as shown in this paper but also the numerical approach based on the recent convergent algorithms for the DUE assignment (see Akamatsu (1998)). Finally, we should explore the case where physical queues are explicitly incorporated into the analysis. Though the incorporation of physical queues may cause very complex phenomena as shown in Daganzo(1998), comprehensive studies on this topic would be indispensable for a clear understanding of the properties of dynamic network flows.

Acknowledgements

The authors gratefully acknowledge stimulating discussions with Nozomu Takamatsu on the topic of this paper. Thanks are also due to Benjamin Heydecker and three anonymous referees for their helpful comments and suggestions.

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Appendix

In Chapter 4 it is assumed that all the OD pairs have always positive flows during the period of analysis. In this appendix, we briefly demonstrate how the formulation can be extended to the case where some OD pairs have no OD flows. The formulations for the one-to-many OD problem and the many-to-one are presented in turn.

(1) One-to-Many OD pattern

We first divide the node set N (where the origin is excluded as a reference node) into two sub sets: the set of destinations with positive OD flows, N_1, and the set of the other destinations, N_2. Then, we divide **A**, $\mathbf{A}_-$, $d\boldsymbol{\tau}(s)/ds$ and $d\mathbf{Q}(s)/ds$ into the two blocks corresponding to N_1 and N_2, respectively:

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}_1\\ \mathbf{A}_2\end{bmatrix},\quad \mathbf{A}_-=\begin{bmatrix}\mathbf{A}_{1-}\\ \mathbf{A}_{2-}\end{bmatrix},\quad \frac{d\boldsymbol{\tau}(s)}{ds}=\begin{bmatrix}\frac{d\boldsymbol{\tau}_1(s)}{ds}\\ \frac{d\boldsymbol{\tau}_2(s)}{ds}\end{bmatrix},\quad \frac{d\mathbf{Q}(s)}{ds}=\begin{bmatrix}\frac{d\mathbf{Q}_1(s)}{ds}\\ \frac{d\mathbf{Q}_2(s)}{ds}\end{bmatrix}=\begin{bmatrix}\frac{d\mathbf{Q}_1(s)}{ds}\\ \mathbf{0}\end{bmatrix} \tag{A-1}$$

For the destinations with positive OD flows, the arrival times are governed by the departure-time equilibrium condition (4.5):

$$\frac{d\boldsymbol{\tau}_1(s)}{ds}=\mathbf{E}\frac{a_1}{a_1-a_2} \tag{A-2}$$

For the other destination nodes, the arrival times should be determined from the route choice equilibrium condition (3.6). Rewriting the condition (3.6) with the variables defined above,

$$\begin{bmatrix}\frac{d\mathbf{Q}_1(s)}{ds}\\ \mathbf{0}\end{bmatrix}=\begin{bmatrix}\mathbf{A}_1\\ \mathbf{A}_2\end{bmatrix}\mathbf{M}\begin{bmatrix}\mathbf{A}_{1-}^T & \mathbf{A}_{2-}^T\end{bmatrix}\begin{bmatrix}\frac{d\boldsymbol{\tau}_1(s)}{ds}\\ \frac{d\boldsymbol{\tau}_2(s)}{ds}\end{bmatrix} \tag{A-3}$$

or equivalently,

$$\frac{d\mathbf{Q}_1(s)}{ds}=\mathbf{A}_1\mathbf{M}\mathbf{A}_{1-}^T\frac{d\boldsymbol{\tau}_1(s)}{ds}+\mathbf{A}_1\mathbf{M}\mathbf{A}_{2-}^T\frac{d\boldsymbol{\tau}_2(s)}{ds} \tag{A-4a}$$

$$\mathbf{0}=\mathbf{A}_2\mathbf{M}\mathbf{A}_{1-}^T\frac{d\boldsymbol{\tau}_1(s)}{ds}+\mathbf{A}_2\mathbf{M}\mathbf{A}_{2-}^T\frac{d\boldsymbol{\tau}_2(s)}{ds} \tag{A-4b}$$

Thus, the elastic DUE conditions are represented as the following system of differential equations:

$$\begin{cases}\frac{d\boldsymbol{\tau}_1(s)}{ds}=\mathbf{E}\frac{a_1}{a_1-a_2},\quad \frac{d\boldsymbol{\tau}_2(s)}{ds}=-\left(\mathbf{A}_2\mathbf{M}\mathbf{A}_{2-}^T\right)^{-1}\left(\mathbf{A}_2\mathbf{M}\mathbf{A}_{1-}^T\right)\mathbf{E}\frac{a_1}{a_1-a_2},\\ \frac{d\mathbf{Q}_1(s)}{ds}=\left\{\left(\mathbf{A}_1\mathbf{M}\mathbf{A}_{1-}^T\right)-\left(\mathbf{A}_1\mathbf{M}\mathbf{A}_{2-}^T\right)\left(\mathbf{A}_2\mathbf{M}\mathbf{A}_{2-}^T\right)^{-1}\left(\mathbf{A}_2\mathbf{M}\mathbf{A}_{1-}^T\right)\right\}\mathbf{E}\frac{a_1}{a_1-a_2}\end{cases} \tag{A-5}$$

(2) Many-to-One OD pattern

We first divide the node set N into two sub sets: the set of origins with positive OD flows, N_1, and the set of the other nodes (including the destination), N_2. Then, we divide **A**, **A_**, $d\boldsymbol{\tau}(u)/du$ and $d\mathbf{Q}(u)/du$ into the two blocks corresponding to N_1 and N_2, respectively:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}, \quad \mathbf{A}_- = \begin{bmatrix} \mathbf{A}_{1-} \\ \mathbf{A}_{2-} \end{bmatrix}, \quad \frac{d\boldsymbol{\tau}(u)}{du} = \begin{bmatrix} \dfrac{d\boldsymbol{\tau}_1(u)}{du} \\ \dfrac{d\boldsymbol{\tau}_2(u)}{du} \end{bmatrix}, \quad \frac{d\mathbf{Q}(u)}{du} = \begin{bmatrix} \dfrac{d\mathbf{Q}_1(u)}{du} \\ \dfrac{d\mathbf{Q}_2(u)}{du} \end{bmatrix}, \tag{A-6}$$

where $d\mathbf{Q}_1(u)/du$ is a column vector with element $dQ_{od}(u)/du$, $d\mathbf{Q}_2/du$ is a column vector whose element is $\mu_d = \sum_k \bar{\mu}_{kd}$ if it corresponding to the destination, otherwise zero.

For the origins with positive OD flows, the departure times are governed by the departure-time equilibrium condition (4.18):

$$\frac{d\boldsymbol{\tau}_1(u)}{du} = \mathbf{E}\frac{a_1 - a_2}{a_1} \tag{A-7}$$

For the other destination nodes, the arrival/departure times should be determined from the route choice equilibrium condition (3.12). Rewriting the condition (3.12) with the variables defined above,

$$\begin{bmatrix} \dfrac{d\mathbf{Q}_1(u)}{du} \\ \dfrac{d\mathbf{Q}_2(u)}{du} \end{bmatrix} = -\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \mathbf{M} \begin{bmatrix} \mathbf{A}_{1-}^T & \mathbf{A}_{2-}^T \end{bmatrix} \begin{bmatrix} \dfrac{d\boldsymbol{\tau}_1(u)}{du} \\ \dfrac{d\boldsymbol{\tau}_2(u)}{du} \end{bmatrix} \tag{A-8}$$

or equivalently,

$$\frac{d\mathbf{Q}_1(u)}{du} = -\left(\mathbf{A}_1\mathbf{M}\mathbf{A}_{1-}^T\right)\mathbf{E}\frac{a_1 - a_2}{a_1} - \left(\mathbf{A}_1\mathbf{M}\mathbf{A}_{2-}^T\right)\frac{d\boldsymbol{\tau}_2(u)}{du} \tag{A-9a}$$

$$\frac{d\mathbf{Q}_2(u)}{du} = -\left(\mathbf{A}_2\mathbf{M}\mathbf{A}_{1-}^T\right)\mathbf{E}\frac{a_1 - a_2}{a_1} - \left(\mathbf{A}_2\mathbf{M}\mathbf{A}_{2-}^T\right)\frac{d\boldsymbol{\tau}_2(u)}{du} \tag{A-9b}$$

Thus, the elastic DUE conditions are represented as the following system of differential equations:

$$\begin{cases} \dfrac{d\boldsymbol{\tau}_1(u)}{du} = \mathbf{E}\dfrac{a_1 - a_2}{a_1}, \quad \dfrac{d\boldsymbol{\tau}_2(u)}{du} = -\left(\mathbf{A}_2\mathbf{M}\mathbf{A}_{2-}^T\right)^T\left\{\left(\mathbf{A}_2\mathbf{M}\mathbf{A}_{1-}^T\right)\mathbf{E}\dfrac{a_1 - a_2}{a_1} + \dfrac{d\mathbf{Q}_2(u)}{du}\right\}, \\ \dfrac{d\mathbf{Q}_1(u)}{du} = -\left\{\left(\mathbf{A}_1\mathbf{M}\mathbf{A}_{1-}^T\right) - \left(\mathbf{A}_1\mathbf{M}\mathbf{A}_{2-}^T\right)\left(\mathbf{A}_2\mathbf{M}\mathbf{A}_{2-}^T\right)^{-1}\left(\mathbf{A}_2\mathbf{M}\mathbf{A}_{1-}^T\right)\right\}\mathbf{E}\dfrac{a_1 - a_2}{a_1} \\ \qquad + \left(\mathbf{A}_1\mathbf{M}\mathbf{A}_{2-}^T\right)\left(\mathbf{A}_2\mathbf{M}\mathbf{A}_{2-}^T\right)^{-1}\dfrac{d\mathbf{Q}_2(u)}{du}. \end{cases} \tag{A-10}$$