A dynamic traffic equilibrium assignment paradox

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Abstract

This paper explores the properties of dynamic flow patterns on two symmetrical networks: an “evening-rush-hour” network (E-net) with a one-to-many origin–destination (OD) pattern and a “morning-rush-hour” network (M-net) with a many-to-one OD pattern that can be obtained by reversing the direction of links and the origin and destinations of the evening-rush-hour network. Although conventional static traffic assignment produces exactly the same flow pattern for both networks, such a simple result does not hold for dynamic assignment. We show this result theoretically by using dynamic equilibrium assignment with a point queue model. Specifically, we first derive the closed form solutions of the dynamic assignment for both networks. We then identify the essential differences between the two dynamic network flow patterns by comparing the mathematical structure of the solutions. Furthermore, a type of capacity paradox (a dynamic version of Braess’s paradox [Braess, D., 1968. Über ein paradox in der verkehrsplanung. Unternehmensforschung, 12, 258–268.]) is identified in order to demonstrate the differences. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Static traffic assignment theory has been the basic framework not only for estimating traffic demands but also for theoretically considering the problems of transportation demand management policies such as congestion tolls. However, when we consider these problems in the framework of dynamic assignment, taking into account the effect of traffic queues, some theoretical conclusions can differ significantly from those derived from the conventional framework of static assignment.

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Consider the two symmetrical networks shown in Fig. 1. The network in Fig. 1(a) has a single origin and multiple destinations. We regard it as an “evening-rush-hour” on a network of a city with a single CBD, and shall refer to it as an “E-net” hereafter. The network in Fig. 1(b) is the “reverse”, with a single destination and multiple origins. We can obtain it by reversing the direction of all the links and origin/destinations of the E-net. Moreover, we may regard it as a “morning-rush-hour” on the same network and shall refer to it as an “M-net”. The standard static traffic assignment produces exactly the same flow patterns for both networks.\(^1\) This is because each path in an M-net has the same cost function as that of the reverse path in the corresponding E-net when the standard assumptions of static traffic assignment are employed. Does this simple conclusion also hold for dynamic assignment? The answer is “no”. Indeed, there seem to be essential differences between the two flow patterns for E-nets and M-nets in the dynamic assignment case.

The purpose of this paper is to explore the differences between the two dynamic flow patterns for E-nets and M-nets. Specifically, we first derive explicit solutions of the dynamic equilibrium assignment on “saturated networks” (the rigorous definition is explained later) with one-to-many origin–destination (OD) patterns. We then identify the differences in the properties of the two dynamic flow patterns by comparing the structure with that of the reverse network. Furthermore, we discuss a particular type of capacity-increasing paradox (a dynamic version of Braess’s paradox) as an example that demonstrates the essential differences between the two flow patterns.

In Section 2, we briefly review the dynamic user equilibrium assignment, restricting ourselves to the minimum knowledge required for considering our problem. Section 3 compares the structure of the dynamic equilibrium assignment problems for E-nets and M-nets. Section 4 considers the dynamic version of Braess’s paradox, and Section 5 summarizes the results and remarks on further research topics.

2. Decomposed formulation of dynamic equilibrium assignment

2.1. Networks and notation

Our model is defined on a transportation network \( G[N, L, W] \) consisting of the set \( L \) of directed links with \( L \) elements, the set \( N \) of nodes with \( N \) elements, and the set \( W \) of OD node pairs. The origins and the destinations are subsets of \( N \), and we denote them by \( R \) and \( S \), respectively. In this paper, we deal only with networks with a one-to-many OD (i.e., the element of \( R \) is unique) or those with a many-to-one OD (i.e., the element of \( S \) is unique). Sequential integer numbers from 1 to \( N \) are allocated to \( N \) nodes. A link from node \( i \) to \( j \) is denoted as link \((i, j)\). We also use a notation to indicate a link by allocating the sequential numbers from 1 to \( L \) to all the links in the set \( L \). The set of upstream nodes of links arriving at node \( k \) is denoted by \( I_k \). Similarly, the set of downstream nodes of links leaving node \( k \) is denoted by \( O_k \).

\(^1\) Strictly speaking, this statement applies to static assignment with “separable” link costs (link performance functions), i.e., where the link cost does not depend on flows on other links; it is not necessarily true for assignment with an asymmetric link cost Jacobian. In many practical cases, however, it seems that setting appropriate link cost functions with link interactions is difficult under the static framework, and that the “standard/conventional static traffic assignment” means assignment with separable link costs.
The structure of a network is represented by a node-link incidence matrix $A^*$, which is an $N$ by $L$ matrix whose $(n, a)$th element is 1 if node $n$ is an upstream-node of link $a$, -1 if node $n$ is a downstream-node of link $a$, and zero otherwise. The rank of this matrix is $N - 1$ since the sum of rows in each column is always zero. Hence, it is convenient for representing our model to use the reduced incidence matrix $A'$ (instead of $A^*$), which is an $(N - 1)$ by $L$ matrix, thus eliminating an arbitrary row of $A^*$. We call the node corresponding to the elimination a "reference node".

2.2. Link model and dynamic equilibrium assignment

For a link model in our dynamic assignment, we employ a First-In-First-Out (FIFO) principle and the point queue concept in which a vehicle has no physical length. It is assumed that the arrival flow at link $(i, j)$ leaves the link after the free flow travel time $m_{ij}$ if there is no queue on the link, otherwise it leaves the link at the maximum departure rate $l_{ij}$. As shown in Akamatsu and Kuwahara (1994) and Daganzo (1995), this model can be represented by subdividing each link into two: one part corresponds to a link with no queuing and length $m_{ij}$, leading into the other part, which has zero-length but infinite queuing space (having a "vertical" queue).

Concerning the assignment principle, we assume the dynamic user equilibrium (DUE) assignment, which is a natural extension of the static user equilibrium assignment. The DUE is defined as the state where no user can reduce his/her travel time by changing his/her route unilaterally for an arbitrary time period (see Kuwahara and Akamatsu, 1993; Akamatsu and Kuwahara, 1994; Smith, 1993; Heydecker and Addison, 1996).

2.3. Basic properties of dynamic equilibrium assignment

As shown by Kuwahara and Akamatsu (1993), under the DUE state, users who depart from their origins at the same time, regardless of their routes, have the same arrival time at any node that they pass through in common on the way to their destination. Furthermore, under the DUE state the order of departure from the origin must be kept at any node through the destinations. From this property, we can define a unique equilibrium arrival time at each node for each departure time from the origin.

Under the link model defined in Section 2.2, the travel time of link $(i, j)$ at time $t$, $C_{ij}(t)$, depends only on the vehicles that arrived at the link before time $t$. Using this and the above discussion on the order of arrivals at a node, it is concluded that the travel time experienced by a
vehicle that departs from an origin at time \( s \) is independent of the flow of vehicles that depart from the origin after time \( s \). Consequently, we can consider the assignment sequentially in the order of departure from a single origin. That is, the assignment can be decomposed with respect to the departure time from a single origin provided the OD pattern is one-to-many. Similarly, for a many-to-one OD pattern, we can easily conclude that the assignment can be decomposed with respect to the arrival time at a single destination.

In order to describe the physical conditions (e.g., FIFO condition at each link and flow conservation at each node) that dynamic network flows should satisfy, variables are generally defined for each link \((i, j)\) at time \( t \), such as cumulative arrivals, \( A_{ij}(t) \), cumulative departures, \( L_{ij}(t) \), number of vehicles (queue length), \( X_{ij}(t) \) or their derivatives. For the DUE assignment, however, a more concise expression is possible by using variables suitable for the arrival/departure time decomposition. We briefly summarize the resulting formulation below (for further details, see Kuwahara and Akamatsu, 1993; Akamatsu and Kuwahara, 1994; Akamatsu, 1996).

### 2.4. Decomposed formulation by origin departure times: one-to-many OD patterns

For the formulation of the DUE assignment on networks with a one-to-many OD pattern, decomposing with respect to the departure time from a single origin is a natural strategy. In the decomposed formulation with origin departure time \( s \), two kinds of variables \((y_{ij}^s, s_i)\) play a central role. The earliest arrival time at node \( i \) is represented by \( s_i \) for a vehicle departing from origin \( o \) at time \( s \). The minimum travel time from the origin to node \( i \) is represented by \( \hat{s}_i \). The link flow rate with respect to \( s \) is given by \( y_{ij}^s \), that is \( y_{ij}^s \equiv \frac{dA_{ij}(s_i)}{ds} \). In addition, we denote the number of vehicles with destination \( d \) departing from origin \( o \) until time \( s \) (cumulative OD demand by departure-time) by \( Q_{od}(s) \).

It may be worth noting that \( y_{ij}^s \) is not the link flow rate in the usual sense. The standard flow rate in traffic engineering is defined as \( \lambda_{ij}(\tau_i) \equiv \frac{dA_{ij}(\tau_i)}{d\tau_i} \). The relationship between the two flow rates is given by

\[
y_{ij}^s = \lambda_{ij}(\tau_i) \frac{d\tau_i}{ds}.
\]

The formulation of the DUE assignment decomposed with respect to the origin departure time is summarized as follows:

(a) **Link travel time functions:** We denote the travel time of link \((i, j)\) for users with origin departure time \( s \) (i.e., \( c_{ij}^s \equiv C_{ij}(\tau_i^s) \)). In the DUE state (under the assumptions of the point queue concept and the FIFO principle), the rate of change in the travel time of link \((i, j)\), with respect to the origin departure time \( s \), \( dc_{ij}^s/ds \), can be represented as a function of \( y_{ij}^s \) and \( \tau_i^s \) (see Fig. 2.):

\[
\frac{dc_{ij}^s}{ds} = \begin{cases} 1 \frac{dX_{ij}(\tau_i^s)}{d\tau_i^s} = \frac{y_{ij}^s}{\mu_{ij}} - \frac{dr_i^s}{ds} & \text{if there is a queue,} \\ 0 & \text{otherwise,} \end{cases}
\]

By integrating Eq. (2.2a) with a given initial link travel time, we obtain the link travel time:

\[
c_{ij}^s = \int_0^s \frac{dc_{ij}^s}{dt} dt + c_{ij}^{s=0} \quad \forall (i,j) \in L \quad \forall s.
\]
(b) **Minimum path choice conditions:** In the DUE state, a user chooses a route with a travel time *(ex post)* that is a minimum over the network. This condition is written as

\[
\begin{align*}
\text{cs}^s_{ij} & \geq \text{cs}^s_{ij} \quad \text{if} \quad y^s_{ij} > 0 \\
\text{cs}^s_{ij} & > \text{cs}^s_{ij} \quad \text{if} \quad y^s_{ij} = 0 \\
\end{align*}
\]

or equivalently,

\[
\begin{align*}
\text{cs}^s_{ij} + \tau^s_i - \tau^s_j & = 0 \\
\text{cs}^s_{ij} + \tau^s_i - \tau^s_j & > 0, \quad y^s_{ij} \geq 0 \\
\end{align*}
\]

\[\forall ij \in L \quad \forall s,\]

(c) **Flow constraints:** In the decomposed DUE formulation, the flow constraints that consist of the FIFO condition for each link and the flow conservation at each node over a network reduce to the following equations (for details see Kuwahara and Akamatsu, 1993; Akamatsu and Kuwahara, 1994):

\[
\sum_{k \in L} y^s_{ik} - \sum_{j \in O_k} y^s_{kj} - \frac{dQ_{od}(s)}{ds} = 0 \quad \forall k \in N, k \neq o \quad \forall s.
\]

(2.5a)

2.5. **Decomposed formulation by destination arrival times: many-to-one OD patterns**

For the networks with a many-to-one OD pattern, by decomposing with respect to the arrival time at a single destination, the discussion almost parallels that of the previous Section 2.4. In the decomposed formulation with destination arrival time *u*, two kinds of variables \((y^u_{ij}, \tau^u_i)\) play a central role. The latest departure time from node *i* is represented by \(\tau^u_i\) for a vehicle reaching destination *d* at time *u*. The minimum travel time from node *i* to the destination is represented by \(\tau^u_i \equiv u - \tau^u_i\). The link flow rate with respect to *u* is given by \(y^u_{ij}\), that is \(y^u_{ij} \equiv dA_{ij}(\tau^u_i) / du\). In addition, we denote the number of vehicles with origin *o* arriving at destination *d* until time *u* (cumulative OD demand by arrival-time) by \(Q_{od}(u)\).

As in Section 2.4, the formulation of the dynamic equilibrium assignment decomposed with respect to the destination arrival-time is summarized as follows:

![Fig. 2. Link travel time.](image-url)
(a) Link travel time functions:
\[
e_{ij}^u = \int_0^u \frac{dc_{ij}^t}{dt} dt + c_{ij}^{u=0} \quad \forall (i, j) \in L \quad \forall s,
\]
where
\[
\frac{dc_{ij}^u}{du} = \begin{cases} \frac{1}{\mu_{ij}} \frac{dX_{ij}(\tau_i^u)}{ds} - \frac{d\tau_i^u}{du} & \text{if there is a queue,} \\ 0 & \text{otherwise.} \end{cases}
\]

(b) Minimum path choice conditions:
\[
\begin{cases} y_{ij}^u \cdot \{c_{ij}^u - \tau_j^u + \tau_i^u\} = 0 \\ c_{ij}^u - \tau_j^u + \tau_i^u \geq 0, \quad y_{ij}^u \geq 0 
\end{cases} \quad \forall ij \in L \quad \forall u
\]

(c) Flow constraints:
\[
\sum_{j \in \alpha_k^u} y_{ij}^u - \sum_{k \in \delta_k^u} y_{ik}^u - \frac{dQ_{kd}(u)}{du} = 0 \quad \forall k \in N, k \neq d \quad \forall u.
\]

3. Equilibrium flow patterns on saturated networks

In general, it is impossible to obtain an analytical solution of the DUE assignment formulated in Section 2. Therefore, exploring the properties of the DUE assignment under general settings is not appropriate for our purpose. Instead, proceed by assuming “saturated networks”, which enable us to obtain an analytical solution. “Saturated networks” are networks that satisfy the following two conditions:

(a) there are inflows on all links over the network, i.e., \(dA_{ij}(t)/dt > 0\) \(\forall (i, j) \in L, \forall t\),
(b) there are queues on all links over the network, i.e., \(C_{ij}(t) > m_{ij}\) \(\forall (i, j) \in L, \forall t\).

Condition (a) is not very restrictive, since we can construct networks satisfying this condition after knowing the set of links with positive flows. Although condition (b) may not be satisfied in many cases, we nevertheless employ this assumption because it gives us an explicit solution of the DUE assignment (as shown below), which enables us to understand the qualitative properties of interest.

We first show the formulation for the E-net and derive the solution in Section 3.1, and then examine the solution for the M-net in Section 3.2. Finally, the comparison between the E-net and M-net are made in Section 3.3.

3.1. Equilibrium on saturated networks with a one-to-many pattern

3.1.1. Formulation

The DUE assignment on a network with a one-to-many OD pattern can be decomposed with respect to the origin departure-time as mentioned in Section 2. Hence, once we determine the method for solving the equilibrium pattern for one particular departure-time, we can obtain the
equilibrium pattern for whole time periods by repeatedly applying the same procedure for successive departure-times. In the following, we consider the problem of obtaining the equilibrium pattern for vehicles departing from origin \( o \) at time \( s \), assuming that the solutions for vehicles departing before time \( s \) are already given.

The minimum path condition (2.4a) holds in the DUE state. From the assumption that \( y_{ij}^s > 0 \) holds for any links on a saturated network, (2.4a) reduces to \( c_{ij}^s + \tau_i^s - \tau_j^s = 0 \). Since this equation should hold for any \( s \), taking the derivative with respect to \( s \) gives

\[
\frac{dc_{ij}^s}{ds} + \frac{d\tau_i^s}{ds} - \frac{d\tau_j^s}{ds} = 0 \quad \forall s. \tag{3.1}
\]

On the other hand, we see from Eq. (2.2a) that the rate of change in the travel time of link \((i, j)\) with respect to origin departure time \( s \), \( dc_{ij}^s/ds \), when there is a queue, is given by

\[
\frac{dc_{ij}^s}{ds} = \frac{1}{\mu_{ij}} y_{ij}^s - \frac{d\tau_i^s}{ds} \quad \forall s. \tag{3.2}
\]

Substituting (3.2) into (3.1), we obtain

\[
y_{ij}^s = \mu_{ij} \frac{d\tau_j^s}{ds} \quad \forall s. \tag{3.3a}
\]

This implies conservation of vehicles on link \((i, j)\) expressed by \( y_{ij}^s \) and \( \tau_j^s \); the flow of vehicles entering link \((i, j)\) must equal the flow leaving (see Fig. 3.).

Since (3.3a) holds for all the links on a saturated network, it follows that

\[
y(s) = -(MA^T) d\tau(s) / ds \quad \forall s, \tag{3.3b}
\]

where \( A \) is an \((N - 1) \times L\) matrix that can be obtained by letting all the +1 elements of \( A \) equal zero

i.e. the \((n, a)\) element is \(-1\) if link \( a \) is arriving at node \( n \), and zero otherwise, \( M \) is an \( L \times L \) diagonal matrix with elements \( \mu_{ij} \),

\( y(s) \) is an \( L \) dimensional column vector with elements \( y_{ij}^s \),

\( d\tau(s)/ds \) is an \( N - 1 \) dimensional column vector with elements \( d\tau_i^s / ds \).

The link flow, \( y \), should also satisfy the flow constraint (2.5a). Defining \( dQ(s)/ds \) as an \( N - 1 \) dimensional vector with elements \( dQ_{od}(s)/ds \) (given), we can express the constraints by the following vector–matrix form:

Fig. 3. The relationship between inflow and outflow on a queuing link.
\[ Ay(s) = -\frac{dQ(s)}{ds} \quad \forall s. \]  
(3.4)

Combining (3.3b) with (3.4),
\[ (AMA^T) \frac{d\tau(s)}{ds} = \frac{dQ(s)}{ds} \quad \forall s. \]  
(3.5)

Thus, we see that the DUE assignment has a unique solution \((d\tau(s)/ds)\) if the rank of the matrix \(AMA^T\) is \(N - 1\).

### 3.1.2 Solution

The rank of the matrix \(AMA^T\) generally depends on the choice of a reference node. For a network with a one-to-many OD, the rank of \(AMA^T\) can be less than \(N - 1\) when we choose an arbitrary node that is not an origin as the reference node. However, the rank is always \(N - 1\) when an origin is employed as the reference node. Furthermore, since the value of \(d\tau_i(s)/ds\) for an origin node is always one from the definition of \(\tau_i(s)\) (i.e., it is not necessary to include the variable \(d\tau_i(s)/ds\) for an origin node among the unknown variables), it is natural to choose an origin as the reference node. In this way, we obtain the equilibrium solution, \(d\tau(s)/ds\), by the following formula:
\[ \frac{d\tau(s)}{ds} = (AMA^T)^{-1} \frac{dQ(s)}{ds}. \]  
(3.6)

In addition, we can obtain the equilibrium link flow pattern, \(y(s)\), by substituting (3.6) into (3.3).

### 3.2 Equilibrium on saturated networks with a many-to-one pattern

#### 3.2.1 Formulation

The DUE assignment on a network with a many-to-one OD pattern can be decomposed with respect to the destination arrival-time as shown in Section 2. In the following, we consider the problem of obtaining the equilibrium pattern for vehicles arriving at a destination at time \(u\), assuming that the solutions for vehicles arriving before time \(u\) are already given.

Following a similar discussion to that given in Section 3.1, using Eqs. (2.4b) and (2.2b), the following conditions hold for saturated networks in the DUE state:
\[ \frac{dc_{ij}^u}{du} - \frac{d\tau_{ij}^u}{du} + \frac{d\tau_{ij}^u}{du} = 0 \quad \forall u, \]  
(3.7)

\[ \frac{dc_{ij}^u}{du} = \frac{1}{\mu_{ij}} y_{ij}^u - \frac{d\tau_{ij}^u}{du} \quad \forall u. \]  
(3.8)

Hence,
\[ y_{ij}^u = \bar{\mu}_{ij} \frac{d\tau_{ij}^u}{du} \quad \forall u. \]  
(3.9a)
Since Eq. (3.9a) holds for all the links on a saturated network, it follows that
\[
y(u) = - (MA^T) \frac{d\tau(u)}{du} \quad \forall u,
\]
where \(y(u)\) is an L dimensional column vector with elements \(y_{ij}\),
\(d\tau(u)/du\) is an \(N-1\) dimensional column vector with elements \(d\tau_i^u/du\).

The link flow, \(y\), should also satisfy the flow constraint (2.5b). Defining \(dQ(u)/du\) as an \(N-1\) dimensional vector with elements \(dQ_{id}(u)/du\) (given), we can express the constraints by the following vector-matrix form:
\[
Ay(u) = \frac{dQ(u)}{du} \quad \forall u.
\]
Combining (3.9b) with (3.10), we obtain
\[
-(AMA^T) \frac{d\tau(u)}{du} = \frac{dQ(u)}{du} \quad \forall u.
\]
Thus, we see that the DUE assignment has a unique solution \((d\tau(u)/du \text{ and } y(u))\) if the rank of \(AMA^T\) is \(N-1\).

3.2.2. Solution

An arbitrary network with a many-to-one OD pattern can be obtained by reversing the direction of all links and the origin/destination of a network with a one-to-many OD pattern. Therefore, it is natural to expect that “reversing” the result in Section 3.1, will lead to the rank of \(AMA^T\) becoming \(N-1\) when a destination is chosen as the reference node. However, this is not the case for this problem. The rank becomes less than \(N-1\) (and therefore, we cannot determine a unique \(d\tau(s)/du\) even if we set the destination as the reference node. Furthermore, we can prove that the rank is less than \(N-1\) for any choice of the reference node.

The rank of the matrix \(AMA^T\) becomes less than \(N-1\) because there are particular origins (we call these “pure origins”) that are not traversal nodes (i.e., an origin that has no arriving links). Letting \(B_{ij}\) be the \((i, j)\) element of \(A^TMA\), we easily see that
\[
B_{ij} = \begin{cases} 
-\bar{\mu}_{ij} & \text{if } i \neq j, \\
\sum_k \mu_{ki} & \text{if } i = j.
\end{cases}
\]
Hence, the column vectors of \(AMA^T\) corresponding to a pure origin are always zero, and the rank of \(AMA^T\) must necessarily decrease by the number of pure origins.

To see this more precisely, we divide the node set \(N\) into two sub-sets: the set of pure origins, \(N_1\), and the set of the other nodes, \(N_2\). Then, we divide \(A\), \(A_\perp\), \(d\tau(u)/du\) and \(dQ(u)/du\) into two blocks corresponding to \(N_1\) and \(N_2\), respectively:
\[
A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad A_\perp = \begin{bmatrix} 0 \\ A_{2\perp} \end{bmatrix}, \quad \frac{d\tau(u)}{du} = \begin{bmatrix} \frac{d\tau_1(u)}{du} \\ \frac{d\tau_2(u)}{du} \end{bmatrix}, \quad \frac{dQ(u)}{du} = \begin{bmatrix} \frac{dQ_1(u)}{du} \\ \frac{dQ_2(u)}{du} \end{bmatrix},
\]
where the \(i\)th element of \(dQ_2(u)/du\) is defined as \(\sum_o \{dQ_{od}(u)/du\} = -\sum_k \mu_{kd} \text{ if } i \text{ is an origin, } dQ_{od}(u)/du \text{ if } i \text{ is a destination, and zero otherwise. Note that } A_{1\perp}, \text{ which is the first block of } A_\perp
corresponding to \( N_1 \), is always 0 according to the definition of pure origins. Rewriting (3.9) with these partitioned variables, we have

\[
\begin{bmatrix}
\frac{dQ_1}{du} \\
\frac{dQ_2}{du}
\end{bmatrix} = -\begin{bmatrix}
A_1 \\
A_2
\end{bmatrix} M \begin{bmatrix}
0 \\
A_{2-}^T
\end{bmatrix} \begin{bmatrix}
\frac{d\tau_1}{du} \\
\frac{d\tau_2}{du}
\end{bmatrix} = \begin{bmatrix}
0 & -A_1M_{2-}^T \\
0 & -A_2M_{2-}^T
\end{bmatrix} \begin{bmatrix}
\frac{d\tau_1}{du} \\
\frac{d\tau_2}{du}
\end{bmatrix}.
\]

(3.12)

That is,

\[
\frac{dQ_1}{du} = -A_1M_{2-}^T \frac{d\tau_1}{du},
\]

(3.13a)

\[
\frac{dQ_2}{du} = -A_2M_{2-}^T \frac{d\tau_2}{du}.
\]

(3.13b)

This means that no condition that determines the \( \frac{d\tau_1}{du} \) for the pure origins is included in the equilibrium condition (3.9), while the \( \frac{d\tau_2}{du} \) for the traversal nodes can be obtained by

\[
\frac{d\tau_2}{du} = -(A_2M_{2-}^T)^{-1} \frac{dQ_2}{du}.
\]

(3.14)

We must add appropriate conditions in order to determine the \( \frac{d\tau_1}{du} \) for the pure origins. An example of a simple and natural way of carrying this out is given by the following: instead of giving the OD flow rates measured at a destination \( q_{od}(u) = dQ_{od}(u)/du \), we give the exogenous OD flow rates measured at origins \( \hat{q}_{od}(u) = dQ_{od}(u)/d\tau_o(u) \) for the pure origins; for other origin nodes, \( dQ_2(u)/du \) (OD flow rates measured at destinations) are given. In this case, we first obtain \( \frac{d\tau_2}{du} \) from Eq. (3.14). Substituting this into Eq. (3.13a), we obtain the OD flow rates for the pure origins, \( q_{od}(u) \), as follows:

\[
\frac{dQ_1}{du} = -(A_1M_{2-}^T) \frac{d\tau_2}{du} = (A_1M_{2-}^T)(A_2M_{2-}^T)^{-1} \frac{dQ_2}{du}.
\]

(3.15)

However, \( q_{od}(u) \) has the following relationship with \( \hat{q}_{od}(u) \):

\[
q_{od}(u) \equiv \frac{dQ_{od}(u)}{du} = \frac{dQ_{od}(u)}{d\tau_o(u)} \frac{d\tau_o(u)}{du} = \hat{q}_{od}(u) \frac{d\tau_o(u)}{du}.
\]

(3.16)

Hence, \( \frac{d\tau_1}{du} \) for the pure origins can be obtained as the ratio of \( q_{od}(u) \) (given by (3.15)) to \( \hat{q}_{od}(u) \) (exogenously given):

\[
\frac{d\tau_1}{du} = \frac{q_{od}(u)}{\hat{q}_{od}(u)}
\]

(3.17a)

or equivalently,

\[
\frac{d\tau_1}{du} = \left(\frac{dQ_1(u)}{d\tau_o(u)}\right)^{-1} (A_1M_{2-}^T)(A_2M_{2-}^T)^{-1} \frac{dQ_2}{du},
\]

(3.17b)
where

\[
\frac{dQ_i(u)}{dt} = \begin{bmatrix}
\vdots & \frac{dQ_{i1}(u)}{dt} & \cdots & 0 \\
0 & \frac{dQ_{i2}(u)}{dt} & \cdots & \vdots 
\end{bmatrix}
\]

3.3. Comparing the two solutions

We have derived the solution of the dynamic equilibrium assignment for both E-nets and M-nets in Sections 3.1 and 3.2, respectively. To consider the differences from a slightly different perspective, let us explicitly compare the two flow patterns on a particular type of E-net and M-net that are identical in structure except that the directions of the links are mutually opposite (e.g., the networks in Fig. 1(a) and (b)).

Let \( A \) be the incidence matrix of the E-net, which can be represented as \( A \equiv A_+ + A_- \). As we have seen in Section 3.1, the equilibrium for the E-net is governed by the following equation:

\[
(AMA^T) \frac{d\tau(s)}{ds} = \frac{dQ(s)}{ds} \quad \forall s.
\] (3.5)

When the E-net has the incidence matrix \( A \equiv A_+ + A_- \), the corresponding M-net has the incidence matrix \( \tilde{A} \equiv \tilde{A}_+ + \tilde{A}_- \) such that \( \tilde{A}_+ = -A_- \) and \( \tilde{A}_- = -A_+ \). Hence, \( \tilde{A}MA^T = AMA^T \) holds. From this and Eq. (3.11) in Section 3.2, we see that the equilibrium for the corresponding M-net is governed by

\[
-(AMA^T) \frac{d\tau(u)}{du} = \frac{dQ(u)}{du} \quad \forall u.
\] (3.11)

Thus, the solutions for the E-net and the corresponding M-net are characterized by the properties of \( AMA^T \) and \( -AMA^T \), respectively. The properties of these two matrices are, in general, very different. They differ in quantitative aspects as well as in the qualitative properties such as rank, as we have seen in Section 3.2. Consequently, the two dynamic flow patterns for E-net and M-net should be significantly different. We shall present a concrete example of such differences in Section 4.

4. Paradoxes

This section presents a discussion of a capacity-increasing paradox as an example that demonstrates the essential difference between the two dynamic flow patterns for E-nets and M-nets. The paradox is a situation where improving the capacity of a certain link on a network worsens the total travel cost over the network. This is a dynamic version of Braess’s paradox (see Murchland (1970)), which is well-known in the analysis of static assignment.\(^2\) Using the results

\(^2\) Strictly speaking, “Braess’s paradox” deals with the situation of adding a new link (not increasing link capacities). Therefore, it may be more appropriate to say that the paradox considered in this paper corresponds to “Smith’s paradox” (Smith, 1978).
obtained in Section 3, we derive the necessary conditions for the occurrence of the paradox for the E-net and M-net, which are shown to be significantly different.

4.1. A paradox for a network with a one-to-many OD pattern

We consider the paradox for the network shown in Fig. 4, where node 1 is a unique origin, nodes 2 and 3 are destinations, and the maximum departure rate (capacity) of link \(a\) \((a = 1, 2, 3)\) is given by \(\mu_a\).

For brevity of notation, we employ the superscript "\(s\)" as the derivative operation with respect to origin departure-time, \(s\), in this section (e.g., \(\dot{t}_i(s) = d\tau_i(s)/ds\), \(\dot{Q}_{od}(s) = dQ_{od}(s)/ds\)).

For the network in Fig. 4, the origin (i.e., node 1) should be the reference node. The incidence matrix \(A\), the reduced incidence matrix \(A_s\), and the corresponding \(A_s^\#\) are given by the following:

\[
A^s = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix}, \quad A_s^\# = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}
\]

Hence,

\[
A^sM^\#A^s_T = \begin{bmatrix} \mu_1 & -\mu_3 \\ 0 & 0 \end{bmatrix}, \quad (A^sM^\#A^s_T)^{-1} = \begin{bmatrix} 1 & \frac{\mu_3}{\mu_1(\mu_2 + \mu_3)} \\ 0 & \frac{1}{\mu_2 + \mu_3} \end{bmatrix}
\]

The equilibrium pattern for the vehicles with departure time \(s\) from a single origin can be calculated using the results of Section 3. From Eq. (3.6), we first obtain the rate of change in equilibrium arrival time:

\[
\dot{t}_2(s) = \frac{1}{\mu_1} \dot{Q}_{12}(s), \quad \dot{t}_3(s) = \frac{1}{\mu_2 + \mu_3} \dot{Q}_{13}(s).
\]

Substituting these into Eq. (3.3), we have the following equilibrium link flow pattern:

\[
y_1(s) = \dot{Q}_{12}(s) + \frac{\mu_3}{\mu_2 + \mu_3} \dot{Q}_{13}(s) = \dot{Q}_{12}(s) + y_3(s), \\
y_2(s) = \frac{\mu_2}{\mu_2 + \mu_3} \dot{Q}_{13}(s), \\
y_3(s) = \frac{\mu_3}{\mu_2 + \mu_3} \dot{Q}_{13}(s).
\]

Fig. 4. Example network with a single origin and two destinations.
In addition, Eq. (3.2) gives the derivatives of equilibrium link costs by the following:

\[
\begin{align*}
\dot{c}_1(s) &= \frac{1}{\mu_1} \dot{Q}_{12}(s) + \frac{\mu_3}{\mu_1(\mu_2 + \mu_3)} \dot{Q}_{13}(s) - 1, \\
\dot{c}_2(s) &= \frac{1}{\mu_2} \dot{Q}_{13}(s) - 1, \\
\dot{c}_3(s) &= \frac{\mu_1 - \mu_3}{\mu_1(\mu_2 + \mu_3)} \dot{Q}_{13}(s) - \frac{1}{\mu_1} \dot{Q}_{12}(s).
\end{align*}
\] (4.5)

To discuss the “capacity increasing paradox”, we employ the total travel time for the users departing from an origin, from time 0 to \(T\), as an indicator of the efficiency of the network flow pattern:

\[
TC \equiv \sum_a \int_0^T y_a(s)c_a(s)ds = \sum_d \int_0^T \dot{Q}_{od}(s)\{\tau_d(s) - s\}ds.
\] (4.6)

We then refer to the situation as a “paradox” if the increase in the capacity of a certain link, \(\mu_a\), results in the increase of \(TC\) (i.e., \(dTC/d\mu_a > 0\) implies “paradox”).

Let us examine whether the paradox arises for the network in Fig. 4. Substituting Eq. (4.3) (or (4.4) and (4.5) into (4.6), we obtain \(TC\):

\[
TC = \int_0^T \left[ \dot{Q}_{12}(s) \left\{ \frac{Q_{12}(s)}{\mu_1} + \frac{\mu_3 Q_{13}(s)}{\mu_1(\mu_2 + \mu_3)} + \tau_2(0) - s \right\} + \dot{Q}_{13}(s) \left\{ \frac{Q_{13}(s)}{\mu_2 + \mu_3} + \tau_3(0) - s \right\} \right] ds.
\] (4.7)

From Eq. (4.7), we easily see that the increase of \(\mu_1\) or \(\mu_2\) always decreases \(TC\) (note that both \(\mu_1\) and \(\mu_2\) appear only in the denominator of \(TC\), and have positive coefficients), that is, the paradox does not arise for links 1 and 2. Increasing \(\mu_3\), however, results in the paradox. The reason for this is that since

\[
\frac{dTC}{d\mu_3} = \left[ \mu_2 \left\{ \int_0^T \dot{Q}_{12}(s)Q_{13}(s)\, ds \right\} - \mu_1 \left\{ \int_0^T \dot{Q}_{13}(s)Q_{13}(s)\, ds \right\} \right] \frac{1}{\mu_1(\mu_2 + \mu_3)^2},
\] (4.8)

if the condition:

\[
\frac{\int_0^T \dot{Q}_{12}(s)Q_{13}(s)\, ds}{\mu_1} > \frac{\int_0^T \dot{Q}_{13}(s)Q_{13}(s)\, ds}{\mu_2}
\] (4.9)

holds, \(dTC/d\mu_3\) is always positive. This means that the paradox has occurred.

The inequality (4.9) is the condition under which the paradox occurs for a certain time period \([0, T]\). From this, we can also derive the condition under which the paradox occurs for an arbitrary time period:

\[
\frac{\dot{Q}_{12}(s)}{\mu_1} > \frac{\dot{Q}_{13}(s)}{\mu_2}.
\] (4.10)

The meaning of this inequality is simple. An increase in \(\mu_3\) always results in an increase in \(y_3\) (see Eq. (4.4)). Suppose there is one unit of increase in flow on link 3 (= \(y_3\)). This means that the number of users with destination 3 who pass through link 1 increases by one unit. The increase in flow on link 1 then causes a \(\dot{Q}_{12}(s)/\mu_1\) increase in total travel time for the users with destination 2 (“User-2”). On the other hand, total travel time for the users with destination 3 (“User-3”)
decreases by \( \dot{Q}_{13}(s) / \mu_2 \), since the flow on link 2 decreases by one unit. Therefore, one unit of increase in flow on link 3 causes the increase of total travel time by \( \dot{Q}_{12}(s) / \mu_1 - \dot{Q}_{13}(s) / \mu_2 \). Thus, Eq. (4.10) gives the condition under which the “net benefit” for User-2 and User-3 (User-3’s benefit minus User-2’s loss) due to the increase of \( \mu_3 \) becomes positive.

### 4.2. A paradox for a network with a many-to-one OD pattern

We consider the paradox for the network in Fig. 5, where node 1 is a unique destination, nodes 2 and 3 are origins, and the maximum departure rate of link \( a \) \( (a = 1, 2, 3) \) is given by \( \mu_a \). For brevity of notation, we employ the superscript “**” as the derivative operation with respect to the destination arrival time \( u \) in this section (e.g., \( \dot{\tau}_i(u) \equiv d\tau_i(u)/du \), \( \dot{Q}_{od}(u) \equiv dQ_{od}(u)/du \)).

Node 3 is the pure origin for the network in Fig. 5. We divide the incidence matrix \( A \), the corresponding \( A_\uparrow \) and the OD flow vector in the following way:

\[
\begin{align*}
A_1 &= \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \text{(node 3),} \\
A_1^- &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \quad dQ_1(u)/du = [\dot{Q}_{31}(u)] \\
A_2 &= \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \text{(node 1),} \\
A_2^- &= \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad dQ_2(u)/du = \begin{bmatrix} -\mu_1 / \mu_2 \\ -\mu_1 + \mu_2 \end{bmatrix}.
\end{align*}
\]

Hence,

\[
(4.11)
\]

\[
A_1^TMA_1^- = \begin{bmatrix} 0 & A_1MA_2^T \\ 0 & A_2MA_2^T \end{bmatrix} = \begin{bmatrix} 0 & -\mu_2 & -\mu_3 \\ 0 & -\mu_1 + \mu_2 & 0 \\ 0 & -\mu_1 & \mu_3 \end{bmatrix}.
\]

\[
(4.12)
\]

The equilibrium pattern for the vehicles with an arrival time \( u \) at a single destination can be calculated from the results of Section 3. From Eq. (3.14), we first obtain the rate of change in equilibrium arrival time for nodes 1 and 2:

\[
\dot{\tau}_1(u) = \frac{\mu_1 + \mu_2}{\mu_1 + \mu_2} = 1, \quad \dot{\tau}_2(u) = \frac{\mu_1 - \dot{Q}_{21}(u)}{\mu_3}.
\]

Substituting these into (3.9) yields the link flow rates (with respect to \( u \))

\[
y_1(u) = \mu_1, \quad y_2(u) = \mu_2, \quad y_3(u) = \mu_1 - \dot{Q}_{21}(u).
\]

\[
(4.13)
\]

\[
(4.14)
\]

Fig. 5. Example network with two origins and a single destination.
Note that this flow pattern is significantly different from that for the reversed network (see Eq. (4.4)). An appropriate condition is required in order to determine the rate of change in the equilibrium arrival time for node 3 (the pure origin). As in the discussion of Section 3.2, we assume, for node 3, that the OD flow rate measured at the origin, \( \dot{q}_{31} = \frac{dQ_{31}(u)}{dt_3(u)} = \frac{Q_{31}(u)}{\tau_3(u)} \), is given. Then the OD flow rate measured at the destination, \( q_{31} = \frac{Q_{31}(u)}{\tau_3(u)} \), is determined from Eq. (3.15) to be
\[
\dot{Q}_{31}(u) = \mu_1 + \mu_2 - \dot{Q}_{21}(u).
\]
Substituting this into (3.17), we obtain the rate of change in equilibrium arrival time at node 3:
\[
\dot{\tau}_3(u) = \frac{\dot{Q}_{31}(u)}{q_{31}(u)} = \frac{\mu_1 + \mu_2 - \dot{Q}_{21}(u)}{q_{31}(u)}.
\]
In addition, (3.8) gives the rate of change in equilibrium cost for each link:
\[
\dot{c}_1(u) = 1 - \frac{\mu_1 - \dot{Q}_{21}(u)}{\mu_3}, \quad \dot{c}_2(u) = 1 - \frac{\mu_1 + \mu_2 - \dot{Q}_{21}(u)}{q_{31}(u)},
\]
\[
\dot{c}_3(u) = \frac{\mu_1 - \dot{Q}_{21}(u)}{\mu_3} - \frac{\mu_1 + \mu_2 - \dot{Q}_{21}(u)}{q_{31}(u)}.
\]
Defining the total travel time for users arriving at a destination, from time 0 to \( T \), as an indicator for measuring the efficiency of the network flow pattern:
\[
TC \equiv \sum_o \int_0^T y_o(u)c_o(u)\, du = \sum_o \int_0^T \dot{Q}_{od}(s)\{u - \tau_o(u)\}\, du,
\]
let us examine whether the paradox arises in the network shown in Fig. 5. Substituting Eqs. (4.13) and (4.16) (or (4.14) and (4.17)) into (4.18), we obtain the \( TC \) for this network
\[
TC = \int_0^T \left[ \dot{Q}_{21}(u) \left\{ u - \frac{\mu_1 u - Q_{21}(u)}{\mu_3} - \tau_2(0) \right\} + \dot{Q}_{31}(u) \left\{ u - \frac{(\mu_1 + \mu_2)u - Q_{21}(u)}{Q_{31}} - \tau_3(0) \right\} \right] du,
\]
where \( \dot{Q}_{31}(u) \equiv \int_0^T \dot{Q}_{od}(u)\, du \). We see from this equation that an increase in \( \mu_1 \) or \( \mu_2 \) will always decrease \( TC \); the paradox does not arise for links 1 and 2. However, an increase in the capacity of link 3 always results in the occurrence of the paradox. This can be easily examined in the following way. Calculating the derivative of \( TC \) with respect to \( \mu_3 \), we have
\[
\frac{dTC}{d\mu_3} = \int_0^T \dot{Q}_{21}(u) \frac{\mu_1 u - Q_{21}(u)}{\mu_3^2} du = \frac{1}{\mu_3} \int_0^T \left\{ \dot{Q}_{21}(u) \int_0^u \dot{\tau}_2(t)\, dt \right\} du.
\]
Note that \( \dot{\tau}_2(u) \) should be positive in the DUE state, because if \( \dot{\tau}_2(u) \) is not positive the users with destination arrival time \( u' > u \) must depart from their origin before the users with arrival time \( u \). This contradicts the assumption that the state is in DUE. Therefore, from Eq. (4.20) and because
the \( \hat{\tau}_2(u) > 0 \) for any \( u \) and the inequality \( dTC/d\mu_3 > 0 \) always holds, we see that the paradox for link 3 takes place without any additional conditions.

As shown in the examples above (Sections 4.1 and 4.2), the paradox arises only for a particular condition for a network with a one-to-many OD pattern, while the corresponding paradox always arises for the reverse network with a many-to-one OD pattern. This is the asymmetrical result that is particular to dynamic traffic assignment.

5. Concluding remarks

This paper showed that, unlike the static assignment framework, the direction of flow plays a significant role in the dynamic assignment case. Specifically, we first derived the closed form solutions of the dynamic equilibrium assignment for a network with a one-to-many OD pattern (E-net) and the reversed network (M-net). We then compared the structure of the solutions and theoretically clarified the essential differences between the two dynamic flow patterns, such as the indeterminacy (non-uniqueness) of the assignment in M-net. Furthermore, we also discussed a capacity increasing paradox as an example that demonstrates the essential difference between the two dynamic flow patterns. The paradox is a situation where improving the capacity of a certain link on a network worsens the total travel cost over the network. Our analysis on a simple network showed that the paradox arises only for a particular condition for E-net, while the corresponding paradox always arises for M-net. This is the asymmetrical result that cannot be seen in the classical static assignment framework and is particular to dynamic assignment.

The conclusions obtained in this paper are based on somewhat restricted assumptions such as a point queue model, dynamic user equilibrium, exogenous OD demands, and “saturated networks”. Nevertheless, it seems plausible that the qualitative conclusion, that is that the properties of the dynamic flow patterns for M-net and E-net are essentially different, should hold for other dynamic traffic assignment models as well. A theoretical examination of this conjecture is an important future research topic.

We discussed the paradox only for a particular simple network. Extending the results to more general settings is also an interesting topic. It may be possible to obtain systematic methods for general networks that detect (without computing the equilibrium patterns) the links where the paradox occurs. We should also explore the case where physical queues are explicitly incorporated into the analysis. Although the incorporation of physical queues may cause very complex phenomena, as shown in Daganzo (1998), comprehensive studies on this topic may be indispensable for a clear understanding of the properties of dynamic network flows.

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