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SELF-ORGANIZATION OF LÖSCH'S HEXAGONS IN ECONOMIC AGGLOMERATION FOR CORE-PERIPHERY MODELS

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Hexagonal population distributions of several sizes are shown to be self-organized from a uniformly inhabited state, which is modeled by a system of places (cities) on a hexagonal lattice. Microeconomic interactions among the places are expressed by a core-periphery model in new economic geography. Lösch's ten smallest hexagonal distributions in central place theory are guaranteed to be existent by equivariant bifurcation analysis on $\mathrm{D}_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n)$, and are obtained by computational analysis. The missing link between central place theory and new economic geography has thus been discovered in light of the bifurcation analysis.

Keywords: Central place theory; core-periphery model; new economic geography; group-theoretic bifurcation theory; hexagons; self-organization.

1. Introduction

In central place theory of economic geography,[1] self-organization of hexagonal market areas of three kinds shown in Fig. 1 was proposed by Christaller [1966] based on market, traffic, and administrative principles. The ten smallest hexagons shown in Fig. 2 were presented as fundamental sizes of market areas by Lösch [1954]. The assemblage of hexagonal market areas with different sizes is expected to produce hierarchical hexagonal distributions of the population of places (cities, towns, villages, etc.).

In economics, a criticism on central place theory was raised that it is not derived from market equilibrium conditions [Fujita *et al.*, 1999, p. 27]. Early studies of the formation of patterns were conducted by Clarke and Wilson [1985], Munz and Weidlich [1990]. Hexagonal distributions, as envisioned with central place theory, were inferred to be

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[1]For books and reviews for central place theory, see, for example [Lösch, 1954; Lloyd & Dicken, 1972; Isard, 1975; Beavon, 1977].

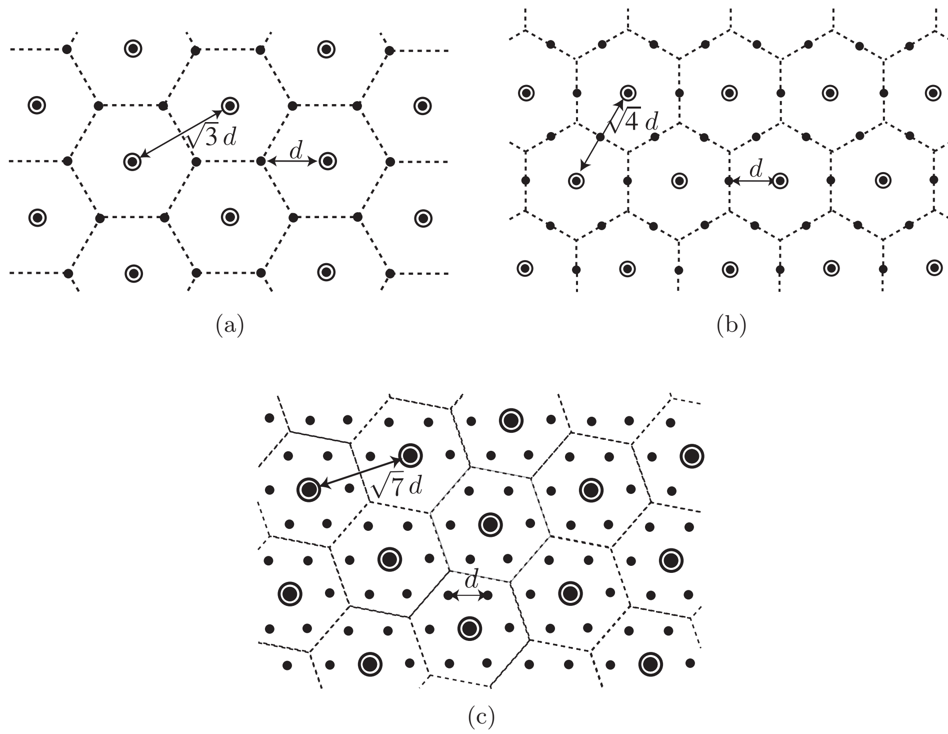

Fig. 1. Three systems predicted by Christaller (the area of a circle indicates the size of population). (a) $k = 3$ system, (b) $k = 4$ system, and (c) $k = 7$ system.

self-organized in core-periphery models in two dimensions by Krugman [1996]. Core-periphery models are capable of expressing the migration of population among cities underpinned by microeconomic mechanism [Krugman, 1991; Combes *et al.*, 2008]. Yet most studies for these models were confined to overly simplified geometry of two-city case.

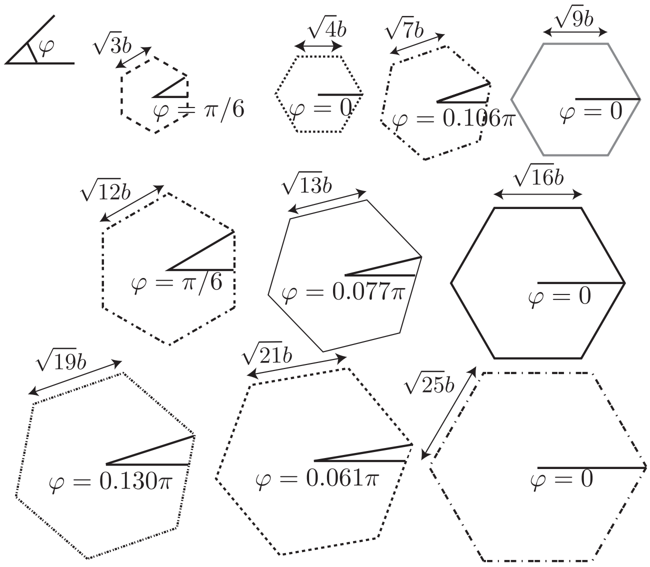

Fig. 2. Lösch's ten smallest hexagons ($b = d/\sqrt{3}$).

To transcend the two-city case, studies on the racetrack economy, which comprises a system of identical cities spread uniformly around the circumference of a circle, have been conducted: Krugman [1993, 1996] conducted local analysis (linearized eigenproblem) of the racetrack economy to identify the emergence of several bifurcating spatial frequencies, Tabuchi and Thisse [2011] have shown the occurrence of spatial period-doubling bifurcation cascade for this economy. The description of this cascade as a hierarchical bifurcation of D_n-symmetric system with $n = 2^m$ is under way [Ikeda *et al.*, 2012].

Hexagonal patterns have been observed for several physical phenomena, including the Bénard problem [Bénard, 1900], and the Faraday experiment [Kudrolli *et al.*, 1998]. The hexagonal patterns in the planar Bénard problem were studied by Sattinger [1978] under a simplifying assumption that solutions are doubly periodic with respect to a hexagonal lattice. Mathematical analysis is conducted on the $\mathrm{D}_6 \dot{+} \mathrm{T}^2$-symmetric hexagonal lattice with periodic boundary conditions [Buzano & Golubitsky, 1983], where D_6 is the dihedral group expressing local hexagonal symmetry and T^2 is the two-torus of translation symmetries. Equivariant bifurcation analysis of six- and twelve-dimensional

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irreducible representations of the group $D_6 \dot{+} T^2$ has been conducted to search for possible bifurcated patterns:

- For six-dimensional ones, hexagons, as well as rolls and triangles, are shown to be existent [Buzano & Golubitsky, 1983; Dionne & Golubitsky, 1992; Golubitsky & Stewart, 2002].
- For twelve-dimensional ones, simple hexagons and super hexagons are shown to be existent [Kirchgässner, 1979; Dionne *et al.*, 1997; Judd & Silber, 2000].

During the course of this, equivariant branching lemma has come to be used as a pertinent means in guaranteeing the existence of a bifurcated solution of a given symmetry [Vanderbauwhede, 1980; Golubitsky *et al.*, 1988]. Nonlinear competition between hexagonal and triangular patterns were studied [Skeldon & Silber, 1998; Silber & Proctor, 1998]. Bifurcated patterns of a honeycomb structure were classified in [Saiki *et al.*, 2005; Ikeda & Murota, 2010, Chapter 16].

The objective of this paper is to demonstrate the self-organization of Lösch's ten smallest hexagons in Fig. 2 for a core-periphery model in two dimensions. It is an important information drawn from the study of the hexagonal patterns by equivariant bifurcation theory that the two-city with D_2-symmetry and the racetrack with D_n-symmetry, which are currently used for the study of core-periphery modes, are insufficient as spatial platforms for the hexagonal distributions for these models. As a pertinent spatial platform, we use a hexagonal lattice with periodic boundaries that comprises uniformly distributed $n \times n$ places that are connected by roads of the same length forming a regular-triangular mesh. The mechanism of microeconomic interactions and migration of workers among the places are expressed by a core-periphery model. The equivariant bifurcation analysis is conducted on a finite group $D_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n)$ that represents the symmetry of uniformly inhabited state of the workers. In comparison with the group $D_6 \dot{+} T^2$, the symmetries of bifurcated solutions of which have been thoroughly obtained in the aforementioned literature, the study of $D_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n)$ poses some additional issues such as the values of n that give the patterns of interest. To be specific, the values of n, the multiplicity of bifurcation points, and irreducible representations corresponding to Lösch's ten smallest hexagons are given and classified. Although there are bifurcation points of various kinds, those which produce the hexagonal patterns are identified and the emergence of those hexagons is successfully demonstrated by computational bifurcation analysis.

This paper is organized as follows: A system of places that is uniformly spread on an infinite hexagonal lattice in two dimensions is modeled in Sec. 2. Section 3 introduces a core-periphery model and predicts its bifurcation mechanism producing hexagonal distributions by group-theoretic bifurcation theory. Group-theoretical prediction of hexagonal distributions for $D_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n)$-symmetric system is carried out in Sec. 4. Computational bifurcation analysis of the $n \times n$ hexagonal lattice is conducted to find bifurcated patterns that represent hexagonal market areas in Sec. 5. Details of the core-periphery model are given in Appendix A. Equivariant bifurcation analysis of twelve-dimensional irreducible representations is carried out in Appendix B.

2. System of Places on a Hexagonal Lattice

We introduce in this section an $n \times n$ hexagonal lattice with periodic boundaries comprising a system of uniformly distributed $n \times n$ places, and prescribe groups expressing the symmetry of this lattice. As a spatial configuration of a system of places, we use the hexagonal lattice because it is geometrically consistent with the hexagonal market areas that are predicted to appear in the literature of economic geography [Lösch, 1954, pp. 133–134].

2.1. *Hexagonal lattice*

Figure 3 portrays the hexagonal lattice, which comprises regular triangles and which covers an infinite two-dimensional domain. A place is allocated at each node of this lattice, expressed by

$$\mathbf{p} = n_1\boldsymbol{\ell}_1 + n_2\boldsymbol{\ell}_2, \quad (n_1, n_2 \in \mathbb{Z}),$$

where $\boldsymbol{\ell}_1 = (d, 0)^\top$ and $\boldsymbol{\ell}_2 = (-d/2, d\sqrt{3}/2)^\top$ are oblique basis vectors (d is the length of these vectors); $\mathbb{Z}$ is the set of integers.

In this paper, we consider a finite $n \times n$ hexagonal lattice with periodic boundary conditions: an example for $n = 2$ is shown by the dashed lines in Fig. 3. A system of $n \times n$ places are allocated at

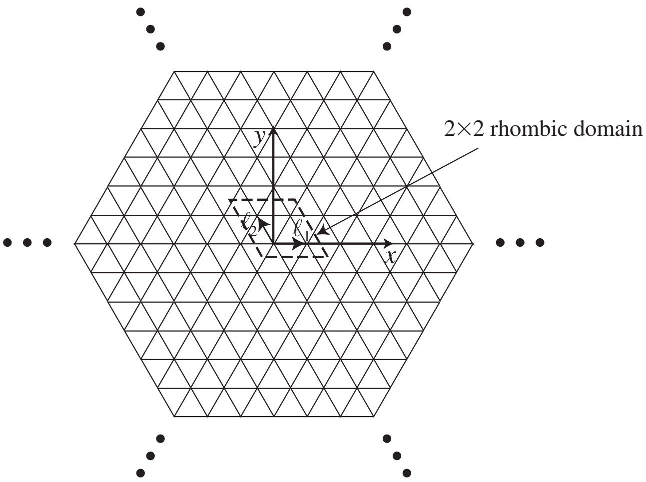

Fig. 3. Hexagonal lattice.

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hexagonal lattice points

$$\mathbf{p} = n_1\boldsymbol{\ell}_1 + n_2\boldsymbol{\ell}_2, \quad (n_1, n_2 = 0, 1, \ldots, n-1)$$

in a finite two-dimensional domain. Neighboring places, in view of the periodic boundaries, are connected by straight lines[2] of equal length d to form a regular-triangular mesh.

2.2. *Two-dimensional periodicity and hexagonal distributions*

If the population distribution of a system of places (i.e. a subset of nodes) has two-dimensional periodicity, then we can set a pair of independent vectors

$$(\mathbf{t}_1, \mathbf{t}_2), \tag{1}$$

called the spatial period vectors, such that the system remains invariant under the translations associated with these vectors. The spatial periods (T_1, T_2) are defined as

$$T_i = \|\mathbf{t}_i\|, \quad (i = 1, 2).$$

The tilted angle φ between $\boldsymbol{\ell}_1$ and $\mathbf{t}_1$ is defined as

$$\cos\varphi = \frac{(\boldsymbol{\ell}_1)^\top \mathbf{t}_1}{\|\mathbf{t}_1\|}. \tag{2}$$

Although the choice of the vectors $(\mathbf{t}_1, \mathbf{t}_2)$ is not unique, T_1 and T_2 must be chosen to be as small as possible, and then to choose the smallest non-negative φ.

To consider hexagonal distributions among possible doubly-periodic distributions, we specifically examine $(\mathbf{t}_1, \mathbf{t}_2)$ of the form

$$\mathbf{t}_1 = \alpha\boldsymbol{\ell}_1 + \beta\boldsymbol{\ell}_2, \quad \mathbf{t}_2 = -\beta\boldsymbol{\ell}_1 + (\alpha - \beta)\boldsymbol{\ell}_2, \quad (\alpha, \beta \in \mathbb{Z}), \tag{3}$$

for which $T_1 = T_2 (\equiv T)$ is satisfied and the angle between $\mathbf{t}_1$ and $\mathbf{t}_2$ is $2\pi/3$. The associated normalized spatial period is given by

$$\frac{T}{d} = \sqrt{\left(\alpha - \frac{\beta}{2}\right)^2 + \left(\frac{\beta\sqrt{3}}{2}\right)^2} = \sqrt{\alpha^2 - \alpha\beta + \beta^2}. \tag{4}$$

We consider a positive integer

$$a = \alpha^2 - \alpha\beta + \beta^2, \tag{5}$$

which can take some specific integer values, such as $1, 3, 4, 7, \ldots$, and rewrite the normalized spatial period in (4) as

$$\frac{T}{d} = \sqrt{a}, \tag{6}$$

which takes some specific values, such as $\sqrt{1}, \sqrt{3}, \sqrt{4}, \sqrt{7}, \ldots$, and lies in the range $1 \le T/d \le n$ in the $n \times n$ system. We refer to the hexagonal distribution with $a = 1$ as the uniform distribution [Fig. 4(a)]. In particular, $a = 3, 4, 7$ correspond respectively to Christaller's $k = 3, 4, 7$ systems [Figs. 4(b)–4(d)]. The values of (α, β) for these systems are given, for example, for Lösch's ten smallest hexagons as listed in Table 1. The tilted angle φ in (2) for the hexagonal distributions is given by

$$\varphi = \arcsin\left(\frac{\dfrac{\beta\sqrt{3}}{2}}{\sqrt{\alpha^2 - \alpha\beta + \beta^2}}\right), \tag{7}$$

and its values are listed in Table 1. With reference to the tilted angle φ defined by (7), we can classify hexagonal distributions into

$$\begin{cases} \text{hexagonal distributions of type V,} \\ \qquad \varphi = 0, \ \ a = 4, 9, 16, 25, \\ \text{hexagonal distributions of type M,} \\ \qquad \varphi = \dfrac{\pi}{6}, \ \ a = 3, 12, \\ \text{tilted hexagonal distributions,} \\ \qquad \text{otherwise}, \ \ a = 7, 13, 19, 21, \end{cases} \tag{8}$$

[2]These straight lines are interpreted as roads in the core-periphery model in Sec. 3.1.

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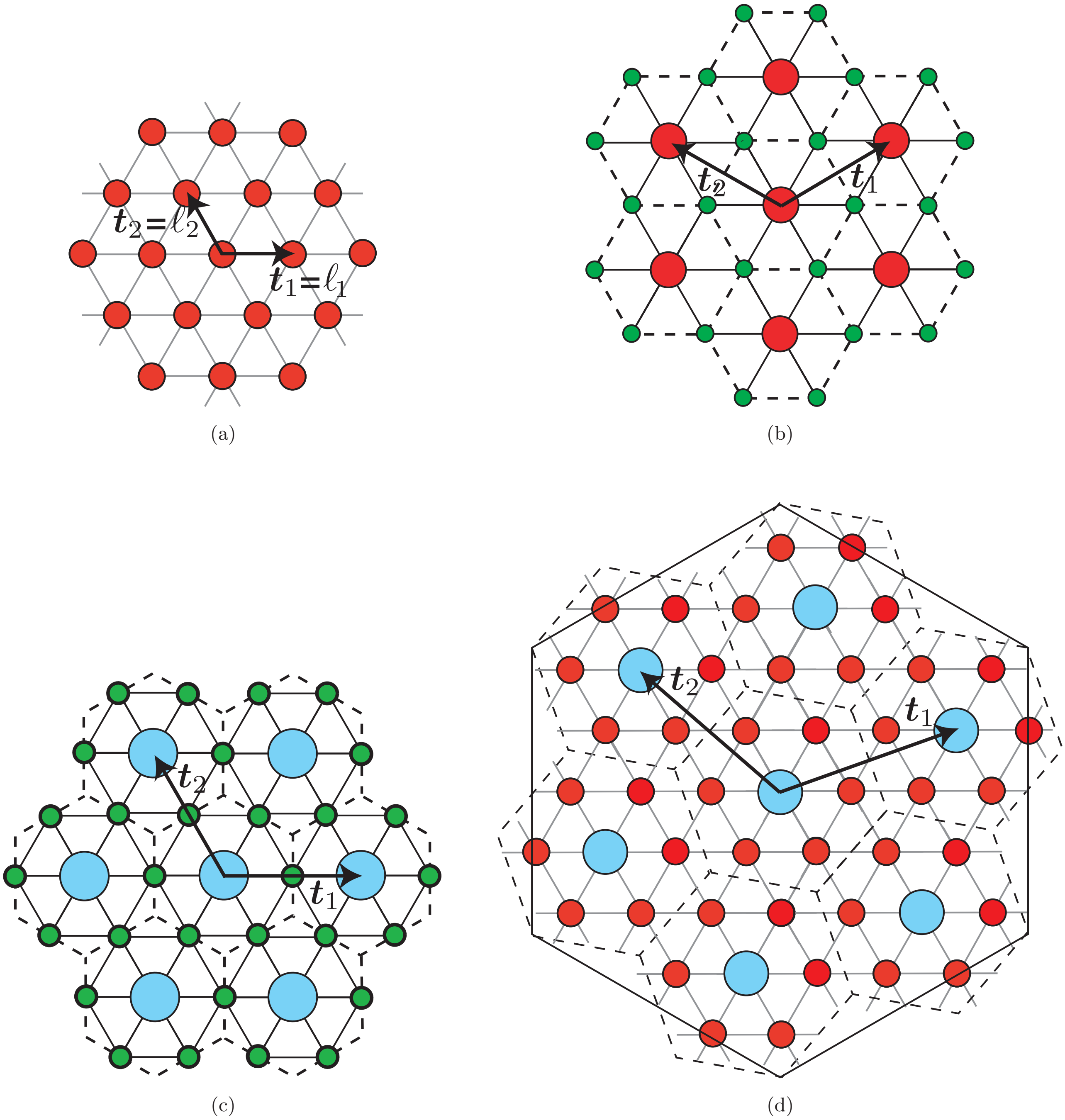

Fig. 4. Hexagonal distributions on the hexagonal lattice. (a) $a = 1$, type V, $(\alpha, \beta) = (1, 1)$, (b) $a = 3$, type M, $(\alpha, \beta) = (2, 1)$ (Christaller's $k = 3$ system) (c) $a = 4$, type V, $(\alpha, \beta) = (2, 0)$ (Christaller's $k = 4$ system) and (d) $a = 7$, tilted, $(\alpha, \beta) = (3, 1)$ (Christaller's $k = 7$ system).

in which "V" signifies that the vertices of the hexagons are located on the x-axis and "M" denotes that midpoints of sides of the hexagons are located on the x-axis. The classification of hexagonal distributions is listed in Table 1. The translational symmetry and the compatible value of n listed in this table are derived later in Sec. 2.3.

2.3. *Groups expressing the symmetry*

For the study of the agglomeration pattern of population distribution on the $n \times n$ hexagonal lattice, we use group-theoretic bifurcation theory: an established mathematical tool for investigating pattern formation. In this theory, the symmetries of possible

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Table 1. The values of (α, β), tilted angle φ, type of hexagon, local and translational symmetries, and compatible n for Lösch's ten smallest hexagons.

a	(α, β)	Tilted Angle φ	Type of Hexagons	Local Symmetry G'_{local}	Translational Symmetry G'_{trans}	Lattice Size n $(m = 1, 2, \ldots)$	M
3	$(2, 1)$	$\pi/6$	M	$\langle r, s\rangle$	$\langle p_1^2 p_2, p_1^{-1} p_2\rangle$	$3m$	2
4	$(2, 0)$	0	V	$\langle r, s\rangle$	$\langle p_1^2, p_2^2\rangle$	$2m$	3
7	$(3, 1)$	0.106π	Tilted	$\langle r\rangle$	$\langle p_1^3 p_2, p_1^{-1} p_2^2\rangle$	$7m$	12
9	$(3, 0)$	0	V	$\langle r, s\rangle$	$\langle p_1^3, p_2^3\rangle$	$3m$	6
12	$(4, 2)$	$\pi/6$	M	$\langle r, s\rangle$	$\langle p_1^4 p_2^2, p_1^{-2} p_2^2\rangle$	$6m$	6
13	$(4, 1)$	0.077π	Tilted	$\langle r\rangle$	$\langle p_1^4 p_2, p_1^{-1} p_2^3\rangle$	$13m$	12
16	$(4, 0)$	0	V	$\langle r, s\rangle$	$\langle p_1^4, p_2^4\rangle$	$4m$	6
19	$(5, 2)$	0.130π	Tilted	$\langle r\rangle$	$\langle p_1^5 p_2^2, p_1^{-2} p_2^3\rangle$	$19m$	12
21	$(5, 1)$	0.061π	Tilted	$\langle r\rangle$	$\langle p_1^5 p_2, p_1^{-1} p_2^4\rangle$	$21m$	12
25	$(5, 0)$	0	V	$\langle r, s\rangle$	$\langle p_1^5, p_2^5\rangle$	$5m$	6

bifurcated solutions are determined with resort to the group that labels the symmetry of the system. Hence the first step of the bifurcation analysis is to identify the underlying group.

2.3.1. *Symmetry of the $n \times n$ hexagonal lattice*

Symmetry of the $n \times n$ hexagonal lattice is characterized by invariance with respect to:

- r: counterclockwise rotation about the origin at an angle of $\pi/3$.
- s: reflection $y \mapsto -y$.
- p_1: periodic translation along the $\boldsymbol{\ell}_1$-axis (i.e. the x-axis).
- p_2: periodic translation along the $\boldsymbol{\ell}_2$-axis.

Consequently, the symmetry of the hexagonal lattice is described by the group

$$G = \langle r, s, p_1, p_2\rangle, \tag{9}$$

where $\langle \cdots \rangle$ denotes a group generated by the elements therein, with the fundamental relations given by

$$r^6 = s^2 = (rs)^2 = p_1^n = p_2^n = e,$$
$$rp_1 = p_1 p_2 r, \quad rp_2 = p_1^{-1} r,$$
$$sp_1 = p_1 s, \quad sp_2 = p_1^{-1} p_2^{-1} s, \quad p_2 p_1 = p_1 p_2,$$

where e is the identity element. Each element of G can be represented uniquely in the form of

$$s^l r^m p_1^i p_2^j, \quad i, j \in \{0, \ldots, n-1\};$$
$$l \in \{0, 1\}; \quad m \in \{0, 1, \ldots, 5\}.$$

(For group theory, see [Curtis & Reiner, 1962; Serre, 1977].)

The group G contains the dihedral group $\langle r, s\rangle \simeq \mathrm{D}_6$ and cyclic groups $\langle p_1\rangle \simeq \mathbb{Z}_n$ and $\langle p_2\rangle \simeq \mathbb{Z}_n$ as its subgroups. Moreover, it has the structure of semidirect product of D_6 by $\mathbb{Z}_n \times \mathbb{Z}_n$, which is denoted as

$$G = \mathrm{D}_6 \dotplus (\mathbb{Z}_n \times \mathbb{Z}_n) \tag{10}$$

or $G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ in another notation. This means, in particular, that $\langle p_1, p_2\rangle$ is a normal subgroup of G.

Remark 2.1. For the group $\mathrm{D}_6 \dotplus \mathrm{T}^2$, where T^2 denotes the two-dimensional torus, a thorough classification of the symmetries of bifurcated solutions has been obtained in the literature using the standard approach based on the equivariant branching lemma [Buzano & Golubitsky, 1983; Dionne *et al.*, 1997]. Naturally, this is closely related to the present study of the bifurcation problem equivariant to $\mathrm{D}_6 \dotplus (\mathbb{Z}_n \times \mathbb{Z}_n)$. Considering the discrete case, with finite n, poses some additional issues. For example, we may be concerned with the values of n that give the patterns of interest, which are important in computational studies.

2.3.2. *Subgroups*

Among many subgroups of $G = \langle r, s, p_1, p_2\rangle = \mathrm{D}_6 \dotplus (\mathbb{Z}_n \times \mathbb{Z}_n)$, we are interested in those subgroups expressing Lösch's ten smallest hexagons. Such subgroups G' are represented as the semidirect product of a subgroup $G'_{\rm local}$ of D_6 by a subgroup $G'_{\rm trans}$ of $\mathbb{Z}_n \times \mathbb{Z}_n$; i.e.

$$G' = G'_{\rm local} \dot{+} G'_{\rm trans}. \tag{11}$$

For example, $G'_{\rm local} = \langle r, s\rangle$ and $G'_{\rm trans} = \langle p_1^2 p_2, p_1^{-1}p_2\rangle$ for the system with $a = 3$. It should be clear that $G'_{\rm local}$ represents the local symmetry and $G'_{\rm trans}$ the translational symmetry.

The local symmetry of the hexagons of Type V or Type M is described by $G'_{\rm local} = \langle r, s\rangle$. The tilted hexagons, lacking in reflection symmetry s, have the local symmetry of $G'_{\rm local} = \langle r\rangle$. Thus the classification of hexagons in (8) gives

$$G'_{\rm local} = \begin{cases} \langle r, s\rangle & \text{for } a = 3, 4, 9, 12, 16, 25, \\ \langle r\rangle & \text{for } a = 7, 13, 19, 21. \end{cases}$$

The translational symmetry is given as

$$G'_{\rm trans} = \langle p_1^{\alpha} p_2^{\beta}, p_1^{-\beta} p_2^{\alpha-\beta}\rangle.$$

Here α and β are the non-negative integers in (3), which are listed in Table 1. From this translational symmetry we can derive a compatibility condition on the size n of the hexagonal lattice for specified a value. For example,

- For $a = 3$ with $(\alpha, \beta) = (2, 1)$, we have $(p_1^2 p_2) \times (p_1^{-1}p_2)^{-1} = p_1^3$, which represents a translation in the direction of the $\boldsymbol{\ell}_1$-axis at the length of $3d$; accordingly, n must be a multiple of 3. The spatial period vectors are given by $(\mathbf{t}_1, \mathbf{t}_2) = (2\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2, -\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2)$. The spatial period elongates as $T/d = 1 \to \sqrt{3}\ (= \sqrt{a})$.
- For $a = 4$ with $(\alpha, \beta) = (2, 0)$, the symmetry of p_1^2 and p_2^2 implies that n is a multiple of 2. The spatial period vectors are given by $(\mathbf{t}_1, \mathbf{t}_2) = (2\boldsymbol{\ell}_1, 2\boldsymbol{\ell}_2)$. The spatial period elongates as $T/d = 1 \to \sqrt{4}\ (= \sqrt{a})$.
- For $a = 7$ with $(\alpha, \beta) = (3, 1)$, we have $(p_1^3 p_2)^2 \times (p_1^{-1}p_2^2)^{-1} = p_1^7$, from which follows that n is a multiple of 7. The spatial period vectors are given by $(\mathbf{t}_1, \mathbf{t}_2) = (3\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2, -\boldsymbol{\ell}_1 + 2\boldsymbol{\ell}_2)$. The spatial period elongates as $T/d = 1 \to \sqrt{7}\ (= \sqrt{a})$.

Likewise, for $a = 9$, 12, 13, 16, 19, 21, 25, respectively, compatible n is a multiple of 3, 6, 13, 4, 19, 21, 5, as listed in Table 1.

Example 2.1. For $n = 3$, the population distribution $\mathbf{h}$ for $a = 3$ is given uniquely as

$$\mathbf{h} = (b, c, c; c, c, b; c, b, c)^{\top}, \tag{12}$$

where $(b, c) = (1/9 + 2\delta, 1/9 - \delta)$ with $-1/18 \le \delta \le 1/9$. This distribution has the symmetry $G' = \langle r, s, p_1^2 p_2, p_1^{-1}p_2\rangle$ with $G'_{\rm local} = \langle r, s\rangle$ and $G'_{\rm trans} = \langle p_1^2 p_2, p_1^{-1}p_2\rangle = \langle p_1^2 p_2\rangle$. The population distribution $\mathbf{h}$ for $n = 3m$ $(m = 2, 3, \ldots)$ can be obtained by spatially repeating the distribution in (12) for $(b, c) = (1/n^2 + 2\delta, 1/n^2 - \delta)$ with $-1/(2n^2) \le \delta \le 1/n^2$.

Example 2.2. For $n = 7$, the population distribution $\mathbf{h}$ for the hexagonal distribution with $a = 7$ is given uniquely as

$$\begin{aligned}\mathbf{h} = (&b, c, c, c, c, c, c;\ c, c, c, b, c, c, c;\ c, c, c, c, c, c, b;\\ &c, c, b, c, c, c, c;\ c, c, c, c, c, b, c;\ c, b, c, c, c, c, c;\\ &c, c, c, c, b, c, c)^{\top},\end{aligned} \tag{13}$$

where $(b, c) = (1/49 + 6\delta, 1/49 - \delta)$ with $-1/294 \le \delta \le 1/49$. This distribution has the symmetry $G' = \langle r, p_1^3 p_2, p_1^{-1}p_2^2\rangle$ with $G'_{\rm local} = \langle r\rangle$ and $G'_{\rm trans} = \langle p_1^3 p_2, p_1^{-1}p_2^2\rangle = \langle p_1^3 p_2\rangle$. The population distribution $\mathbf{h}$ for $n = 7m$ $(m = 2, 3, \ldots)$ can be obtained by spatially repeating the distribution in (12) for $(b, c) = (1/n^2 + 6\delta, 1/n^2 - \delta)$ with $-1/(6n^2) \le \delta \le 1/n^2$.

3. Core-Periphery Model and Bifurcation

In this section, we present a multiregional core-periphery model. The group-equivariance of this model for the system of places is introduced and the mechanism of bifurcation producing hexagonal distributions is studied. Details are given in Appendix A.

3.1. *Core-periphery model*

We employ a core-periphery model by Forslid and Ottaviano [2003] that replaces the production function of Krugman with that of Flam and Helpman [1987].

The economy is composed of K places (labeled $i = 1, \ldots, K$), two factors of production (skilled and unskilled labor), and two sectors (manufacture M and agriculture A). There, H skilled and L unskilled workers consume two final goods: manufactural-sector goods and agricultural-sector goods. Workers supply one unit of each type of labor inelastically; here H is a constant expressing the total number of skilled workers. Skilled workers are mobile across places, and the number of skilled workers in place i is denoted by h_i. Unskilled workers are immobile

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and equally distributed across all places with the unit density (i.e. $L = 1 \times K$). Hence the population in place i is equal to $h_i + 1$.

The governing equation of this model is formulated in a standard form of static equilibrium as

$$\mathbf{F}(\mathbf{h}, \tau) = H\mathbf{P}(\mathbf{h}) - \mathbf{h} = \mathbf{0}. \tag{14}$$

Therein $\mathbf{h} = (h_i) \in \mathbb{R}^K$ is a K-dimensional vector expressing the population distribution of the skilled workers, $\tau \in \mathbb{R}$ is a (bifurcation) parameter corresponding to the transport parameter, and $\mathbf{F}$: $\mathbb{R}^K \times \mathbb{R} \to \mathbb{R}^K$ is a sufficiently smooth nonlinear function in $\mathbf{h}$ and τ; $\mathbf{P} = (P_i) \in \mathbb{R}^K$ is a K-dimensional vector given by

$$P_i(\mathbf{h}, \tau) \equiv \frac{\exp[\theta v_i(\mathbf{h}, \tau; \mu, \sigma)]}{\sum_{j=1}^{K} \exp[\theta v_j(\mathbf{h}, \tau; \mu, \sigma)]}, \quad i = 1, \ldots, K, \tag{15}$$

where θ is the constant representing the inverse of variance of the idiosyncratic tastes, μ is the constant expenditure share on industrial varieties, σ expresses the constant elasticity of substitution between any two varieties, and $v_i(\mathbf{h}, \tau; \mu, \sigma)$ $(i = 1, \ldots, K)$ are nonlinear functions representing the components of an indirect utility function vector $\mathbf{v}(\mathbf{h}, \tau; \mu, \sigma)$.

The equality $H = \sum_{i=1}^{K} h_i$ is satisfied by any solution of (14) because $\sum_{i=1}^{K} P_i(\mathbf{h}, \tau) = 1$ by (15). As a normalization we put $H = 1$ in the subsequent analysis.

3.2. *Exploiting symmetry of core-periphery model*

For investigation of the patterns of the bifurcated solutions, it is crucial to formulate the symmetry that is inherent in the governing equation. In group-theoretic bifurcation theory, the symmetry of the equation for the system of $n \times n$ places on the hexagonal lattice is described as

$$T(g)\mathbf{F}(\mathbf{h}, \tau) = \mathbf{F}(T(g)\mathbf{h}, \tau), \quad g \in G, \tag{16}$$

in terms of an orthogonal matrix representation T of group $G = \langle r, s, p_1, p_2 \rangle$ in (9) on the K-dimensional space $\mathbb{R}^K$. The condition (or property) (16) is called the equivariance of $\mathbf{F}(\mathbf{h}, \tau)$ to G. The most important consequence of the equivariance (16) is that the symmetries of the whole set of possible bifurcated solutions can be obtained and classified.

In our study of a system of $n \times n$ places on the hexagonal lattice, each element g of G acts as a permutation among place numbers $(1, \ldots, K)$ for $K = n^2$ and hence each $T(g)$ is a permutation matrix. Then we can show the equivariance (16) of the core-periphery model to $G = \langle r, s, p_1, p_2 \rangle = \mathrm{D}_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n)$.

Proof. By expressing the action of $g \in G$ as $g : i \mapsto i^*$ for place numbers i and i^*, we have $v_i(T(g)\mathbf{h}, \tau) = v_{i^*}(\mathbf{h}, \tau)$ and $P_i(T(g)\mathbf{h}, \tau) = P_{i^*}(\mathbf{h}, \tau)$ by (15) for any $g \in G$. Therefore, we have

$$\begin{aligned} F_i(T(g)\mathbf{h}, \tau) &= HP_i(T(g)\mathbf{h}, \tau) - h_{i^*} \\ &= HP_{i^*}(\mathbf{h}, \tau) - h_{i^*} \\ &= F_{i^*}(\mathbf{h}, \tau). \end{aligned}$$

This proves the equivariance (16). ■

The group-theoretic bifurcation analysis proceeds as follows. Consider a critical point $(\mathbf{h}_c, \tau_c)$ of multiplicity M (≥ 1), at which the Jacobian matrix of $\mathbf{F}$ has M zero eigenvalues. Throughout this paper we assume that a critical point is generic (or group-theoretic) in the sense that the M-dimensional kernel space of the Jacobian matrix is irreducible with respect to the representation T. See Remark 3.1.

Using a standard procedure called the *Lyapunov–Schmidt reduction with symmetry* [Sattinger, 1979; Golubitsky *et al.*, 1988; Ikeda & Murota, 2010], the full system of equations

$$\mathbf{F}(\mathbf{h}, \tau) = \mathbf{0} \tag{17}$$

in $\mathbf{h} \in \mathbb{R}^K$ [see (14)] is reduced, in a neighborhood of $(\mathbf{h}_c, \tau_c)$, to a system of M equations (called bifurcation equations)

$$\tilde{\mathbf{F}}(\mathbf{w}, \tilde{\tau}) = \mathbf{0} \tag{18}$$

in $\mathbf{w} \in \mathbb{R}^M$, where $\tilde{\mathbf{F}} : \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}^M$ is a function and $\tilde{\tau} = \tau - \tau_c$ denotes the increment of τ. In this reduction process the equivariance of the full system, which is formulated in (16), is inherited by the reduced system (18) in the following form:

$$\tilde{T}(g)\tilde{\mathbf{F}}(\mathbf{w}, \tilde{\tau}) = \tilde{\mathbf{F}}(\tilde{T}(g)\mathbf{w}, \tilde{\tau}), \quad g \in G, \tag{19}$$

where $\tilde{T}$ is the subrepresentation of T on the M-dimensional kernel space of the Jacobian matrix. The symmetry of the kernel space, sometimes referred to as the kernel symmetry, is expressed by the subgroup $\{g \in G \mid \tilde{T}(g) = I\}$. It is this inheritance of symmetry that plays a key role in determining the symmetry of bifurcating solutions.

The reduced equation (18) is to be solved for $\mathbf{w}$ as $\mathbf{w} = \mathbf{w}(\tilde{\tau})$, which is often possible by virtue of the symmetry of $\tilde{\mathbf{F}}$ described in (19). Since $(\mathbf{w}, \tilde{\tau}) = (\mathbf{0}, 0)$ is a singular point of (18), there can be many solutions $\mathbf{w} = \mathbf{w}(\tilde{\tau})$ with $\mathbf{w}(0) = \mathbf{0}$, which give rise to bifurcation. Each $\mathbf{w}$ uniquely determines a solution $\mathbf{h}$ of the full system (17).

The symmetry of $\mathbf{h}$ is represented by a subgroup of G defined by

$$\Sigma(\mathbf{h}) = \Sigma(\mathbf{h}; G, T) = \{g \in G \,|\, T(g)\mathbf{h} = \mathbf{h}\}, \quad (20)$$

called the isotropy subgroup of $\mathbf{h}$. The isotropy subgroup $\Sigma(\mathbf{h})$ can be computed in terms of the symmetry of the corresponding $\mathbf{w}$ as

$$\Sigma(\mathbf{h}; G, T) = \Sigma(\mathbf{w}; G, \tilde{T}), \quad (21)$$

where

$$\Sigma(\mathbf{w}; G, \tilde{T}) = \{g \in G \,|\, \tilde{T}(g)\mathbf{w} = \mathbf{w}\}. \quad (22)$$

The relation (21) enables us to determine the symmetry of bifurcated solutions $\mathbf{h}$ through the analysis of bifurcation equations in $\mathbf{w}$.

Remark 3.1. The number N_d of d-dimensional irreducible representations of $G = \mathrm{D}_6 \dotplus (\mathbb{Z}_n \times \mathbb{Z}_n)$ is given in Table 2(a). Here m is a positive integer. For some values of n (treated in Sec. 5), the numbers N_d of the d-dimensional irreducible representations are listed in Table 2(b).

Table 2. Number N_d of d-dimensional irreducible representations of $\mathrm{D}_6 \dotplus (\mathbb{Z}_n \times \mathbb{Z}_n)$.

(a)

	1	2	3	4	6	12
$n\backslash d$	N_1	N_2	N_3	N_4	N_6	N_{12}
$6m$	4	4	4	1	$2n-6$	$(n^2-6n+12)/12$
$6m \pm 1$	4	2	0	0	$2n-2$	$(n^2-6n+5)/12$
$6m \pm 2$	4	2	4	0	$2n-4$	$(n^2-6n+8)/12$
$6m \pm 3$	4	4	0	1	$2n-4$	$(n^2-6n+9)/12$

(b)

	1	2	3	4	6	12	
$n\backslash d$	N_1	N_2	N_3	N_4	N_6	N_{12}	$\sum N_d$
2	4	2	4				10
3	4	4		1	2		11
4	4	2	4		4		14
5	4	2			8		14
6	4	4	4	1	6	1	20
7	4	2			12	1	19
8	4	2	4		12	2	24
9	4	4		1	14	3	26
10	4	2	4		16	4	30
11	4	2			20	5	31
12	4	4	4	1	18	7	38
13	4	2			24	8	38
14	4	2	4		24	10	44
15	4	4		1	26	12	47
16	4	2	4		28	14	52
17	4	2			32	16	54
18	4	4	4	1	30	19	62
19	4	2			36	21	63
20	4	2	4		36	24	70
21	4	4		1	38	27	74
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
25	4	2			48	40	94

Remark 3.2. Simple bifurcation points do not play a role in the present analysis. The one-dimensional irreducible representations of the group $G = \langle r, s, p_1, p_2 \rangle$ in (9), which we label as $(+,+)$, $(+,-)$, $(-,+)$, and $(-,-)$, are given by

$$\begin{aligned}
T^{(+,+)}(r) &= 1, & T^{(+,+)}(s) &= 1,\\
T^{(+,+)}(p_1) &= 1, & T^{(+,+)}(p_2) &= 1,\\
T^{(+,-)}(r) &= 1, & T^{(+,-)}(s) &= -1,\\
T^{(+,-)}(p_1) &= 1, & T^{(+,-)}(p_2) &= 1,\\
T^{(-,+)}(r) &= -1, & T^{(-,+)}(s) &= 1,\\
T^{(-,+)}(p_1) &= 1, & T^{(-,+)}(p_2) &= 1,\\
T^{(-,-)}(r) &= -1, & T^{(-,-)}(s) &= -1,\\
T^{(-,-)}(p_1) &= 1, & T^{(-,-)}(p_2) &= 1.
\end{aligned}$$

In general $(+,+)$ is associated with a limit point of the bifurcation parameter τ, and $(+,-)$, $(-,+)$, and $(-,-)$ with a simple bifurcation point with $\Sigma(\mathbf{h}) = \langle r, p_1, p_2 \rangle$, $\langle s, p_1, p_2 \rangle$, and $\langle sr, p_1, p_2 \rangle$, respectively. Yet, for the present definition of $\mathbf{h} = (h_i)$ in Sec. 3.1 for the hexagonal lattice with $n \times n$ places, such bifurcation points are nonexistent since $\langle p_1, p_2 \rangle$-symmetry restricts $\mathbf{h}$ to be $G = \langle r, s, p_1, p_2 \rangle$-symmetric, which corresponds to the uniform population. Alternatively, we can say in more technical terms that the irreducible representations $(+,-)$, $(-,+)$, and $(-,-)$ are not contained in the representation $T(g)$ for the core-periphery model.

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4. Theoretically Predicted Hexagonal Distributions

By using group-theoretic bifurcation theory, we present in this section a possible bifurcation mechanism that can produce Lösch's ten smallest hexagons (Sec. 2.2). It is noted first that uniformly distributed population of the skilled workers, given by $h_1 = \cdots = h_{n^2} = 1/n^2$, is the simplest hexagonal distribution associated with the pre-bifurcation solution of the governing equation (14). The symmetry of this solution is labeled by the group

$$G = \langle r, s, p_1, p_2 \rangle = \mathrm{D}_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n)$$

in (9) and (10).

The symmetry of a bifurcated solution $\mathbf{h}$ of the governing equation (14), in general, is expressed by a subgroup $\Sigma(\mathbf{h})$ of G in (20). Among many possible symmetries of bifurcated solutions, we are particularly interested in those bifurcated solutions, if any, for which $\Sigma(\mathbf{h})$ coincides with subgroups in (11) corresponding to Lösch's ten smallest hexagons (Table 1):

$$G'_{\text{local}} \dot{+} G'_{\text{trans}} = \begin{cases} \langle r, s \rangle \dot{+} \langle p_1^2 p_2, p_1^{-1} p_2 \rangle & \text{for } a = 3, \\ \langle r, s \rangle \dot{+} \langle p_1^2, p_2^2 \rangle & \text{for } a = 4, \\ \langle r \rangle \dot{+} \langle p_1^3 p_2, p_1^{-1} p_2^2 \rangle & \text{for } a = 7, \\ \langle r, s \rangle \dot{+} \langle p_1^3, p_2^3 \rangle & \text{for } a = 9, \\ \langle r, s \rangle \dot{+} \langle p_1^4 p_2^2, p_1^{-2} p_2^2 \rangle & \text{for } a = 12, \\ \langle r \rangle \dot{+} \langle p_1^4 p_2, p_1^{-1} p_2^3 \rangle & \text{for } a = 13, \\ \langle r, s \rangle \dot{+} \langle p_1^4, p_2^4 \rangle & \text{for } a = 16, \\ \langle r \rangle \dot{+} \langle p_1^5 p_2^2, p_1^{-2} p_2^3 \rangle & \text{for } a = 19, \\ \langle r \rangle \dot{+} \langle p_1^5 p_2, p_1^{-1} p_2^4 \rangle & \text{for } a = 21, \\ \langle r, s \rangle \dot{+} \langle p_1^5, p_2^5 \rangle & \text{for } a = 25. \end{cases} \tag{23}$$

The main message of this section is that such bifurcated solutions do exist, and therefore Lösch's ten smallest hexagons can be understood within the framework of group-theoretic bifurcation theory. We shall see that Lösch's ten smallest hexagons emerge from bifurcation points of multiplicity $M = 2, 3, 6$, and 12, but not of $M = 1$ and 4. Specifically, we have

$$a = \begin{cases} 3 & \text{for } M = 2, \\ 4 & \text{for } M = 3, \\ 9, 12, 16, 25 & \text{for } M = 6, \\ 7, 13, 19, 21 & \text{for } M = 12. \end{cases}$$

Lösch's hexagons with $a = 9, 12, 16, 25$ for $M = 6$ are called "hexagons" in [Buzano & Golubitsky, 1983] and Lösch's hexagons with $a = 7, 13, 19, 21$ for $M = 12$ are called "simple hexagons" in [Dionne *et al.*, 1997].

Our analysis for specific cases are described below (Secs. 4.2–4.5) and mathematical analysis of the bifurcation equations at group-theoretic bifurcation points of multiplicity $M = 12$ is worked out in Appendix B. The emergence of these hexagons is confirmed numerically in Sec. 5 by the computational bifurcation analysis of the hexagonal lattice with various sizes n.

4.1. *Analysis by equivariant branching lemma*

The emergence of Lösch's hexagons is proved by applying the equivariant branching lemma [Vanderbauwhede, 1980] to the bifurcation equation $\tilde{\mathbf{F}}(\mathbf{w}, \tilde{\tau})$ in (18); see, e.g. [Golubitsky *et al.*, 1988] for this lemma. Recall that bifurcation equation is associated with an irreducible representation of G and that the isotropy subgroup $\Sigma(\mathbf{h})$ in (20) expressing the symmetry of a bifurcated solution $\mathbf{h}$ is identical with the isotropy subgroup $\Sigma(\mathbf{w})$ in (22) of the corresponding solution $\mathbf{w}$ for the bifurcation equation, i.e. $\Sigma(\mathbf{h}) = \Sigma(\mathbf{w})$ as shown in (21). A subgroup Σ is said to be an isotropy subgroup if $\Sigma = \Sigma(\mathbf{h})$ for some $\mathbf{h}$.

The analysis based on the equivariant branching lemma proceeds as follows:

- Specify an isotropy subgroup Σ of G for the symmetry of a possible bifurcated solution as well as an irreducible representation $\tilde{T}$ of G that can possibly be associated with the bifurcation point.
- Obtain the fixed-point subspace $\mathrm{Fix}(\Sigma)$ for the isotropy subgroup Σ with respect to the irreducible representation $\tilde{T}$, where

$$\mathrm{Fix}(\Sigma) = \{\mathbf{w} \in \mathbb{R}^M \mid \tilde{T}(g)\mathbf{w} = \mathbf{w} \text{ for all } g \in \Sigma\}. \tag{24}$$

- Calculate the dimension $\dim \mathrm{Fix}(\Sigma)$ of this subspace.
- If $\dim \mathrm{Fix}(\Sigma) = 1$, a bifurcated solution with symmetry Σ is guaranteed to exist generically by the equivariant branching lemma. If $\dim \mathrm{Fix}(\Sigma) = 0$, a bifurcated solution with symmetry Σ is nonexistent. If $\dim \mathrm{Fix}(\Sigma) \geq 2$, no definite conclusion can be reached by the equivariant branching lemma.

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Isotropy subgroups with $\dim \mathrm{Fix}(\Sigma) = 1$ are called *axial subgroups* and the associated spatially doubly periodic solutions are called *axial planforms* [Golubitsky *et al.*, 1994].

In our present analysis, we employ the above procedure with $\Sigma = G'$ for each G' in (23) and for each irreducible representation $\tilde{T}$ of G; note that each G', representing the symmetry of a Lösch's hexagon, is an isotropy subgroup. Since the dimension of $\tilde{T}$ is either $d = 1, 2, 3, 4, 6$ or 12, the multiplicity M of the critical point is generically either 1, 2, 3, 4, 6 or 12. The equivalent branching lemma applies only if $\dim \mathrm{Fix}(\Sigma) = 1$. Fortunately, it will turn out (see Secs. 4.2–4.4) that, in all cases of our interest in (23), we have $\dim \mathrm{Fix}(\Sigma) \leq 1$ and therefore we can always rely on the equivalent branching lemma to determine the existence or nonexistence of bifurcated solutions for Lösch's ten smallest hexagons.

4.2. *Hexagon with $a = 3$: Bifurcation point of multiplicity 2*

When n is a multiple of 3, hexagons with $a = 3$ appear generically as a branch from a double bifurcation point ($M = 2$) that is associated with the irreducible representation given by

$$T(r) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$T(p_1) = T(p_2) = \begin{bmatrix} \cos\dfrac{2\pi}{3} & -\sin\dfrac{2\pi}{3} \\ \sin\dfrac{2\pi}{3} & \cos\dfrac{2\pi}{3} \end{bmatrix}. \tag{25}$$

This is one of the four two-dimensional irreducible representations of $\mathrm{D}_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ (Table 2).

The general procedure in Sec. 4.1 is applied to

$$\begin{aligned} \Sigma &= \langle r, s, p_1^2 p_2, p_1^{-1} p_2 \rangle \\ &= \langle r, s \rangle \dot{+} \langle p_1^2 p_2, p_1^{-1} p_2 \rangle, \end{aligned} \tag{26}$$

which describes the symmetry of the hexagon with $a = 3$ (Christaller's $k = 3$ system) in Fig. 4(b). The fixed-point subspace $\mathrm{Fix}(\Sigma)$ with respect to $\tilde{T} = T$ in (25) is a one-dimensional subspace of $\mathbb{R}^2$ spanned by $(1, 0)^\top$. Then, by the equivariant branching lemma, there exists a bifurcated path with the symmetry of (26).

It is mentioned that the standard results for a double bifurcation point for the dihedral group symmetry can be adapted to this case with (25). In particular, the concrete form of the bifurcation equations can be determined and the number and the asymptotic form of bifurcated paths can be analyzed; see [Sattinger, 1979; Golubitsky *et al.*, 1988; Ikeda & Murota, 2010, Chapter 8].

Remark 4.1. For $n = 3^m$ (m is a positive integer), there are successive bifurcations associated with a hierarchy of subgroups

$$\begin{aligned} &\mathrm{D}_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n) \to \mathrm{D}_6 \dot{+} \langle p_1^2 p_2, p_1^{-1} p_2 \rangle \\ &\quad \to \mathrm{D}_6 \dot{+} (\mathbb{Z}_{n/3} \times \mathbb{Z}_{n/3}) \\ &\quad \to \cdots \to \mathrm{D}_6 \dot{+} \langle p_1^{2n/3} p_2^{n/3}, p_1^{-n/3} p_2^{n/3} \rangle \\ &\quad \to \mathrm{D}_6 \dot{+} (\mathbb{Z}_1 \times \mathbb{Z}_1) = \mathrm{D}_6, \end{aligned} \tag{27}$$

where $\to$ means the occurrence of bifurcation. These successive bifurcations produce a set of nested hexagons (see computational analysis in Sec. 5.1). The spatial period is multiplied $\sqrt{3}$-times successively as

$$\frac{T}{d} = 1 \to \sqrt{3} \to 3 \to \cdots \to \frac{n}{\sqrt{3}} \to n. \tag{28}$$

This fact can be proved as follows. The subgroup

$$\begin{aligned} \mathrm{D}_6 \dot{+} \langle p_1^2 p_2, p_1^{-1} p_2 \rangle &= \mathrm{D}_6 \dot{+} \langle p_1^2 p_2, p_1^3 \rangle \\ &= \mathrm{D}_6 \dot{+} \langle q_1, q_2 \rangle \end{aligned}$$

($q_1 = p_1^2 p_2, q_2 = p_1^3$) has the two-dimensional irreducible representation

$$T(r) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$T(q_1) = \begin{bmatrix} \cos\dfrac{2\pi}{3} & -\sin\dfrac{2\pi}{3} \\ \sin\dfrac{2\pi}{3} & \cos\dfrac{2\pi}{3} \end{bmatrix}, \quad T(q_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{29}$$

Since the bifurcation equation equivariant with respect to (29) takes the same form as that for the direct bifurcation associated with (25), by the analysis of this equation, we see that

$$\begin{aligned} \Sigma(\mathbf{h}) = \langle r, s, q_1^3, q_2 \rangle &= \langle r, s, p_1^6 p_2^3, p_1^3 \rangle \\ &= \langle r, s, p_1^3, p_2^3 \rangle. \end{aligned} \tag{30}$$

This process is repeated to prove (27).

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4.3. *Hexagon with $a = 4$: Bifurcation point of multiplicity 3*

When n is a multiple of 2, hexagons with $a = 4$ are predicted to branch from a triple bifurcation point that is associated with the three-dimensional irreducible representation given as

$$T(r) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad T(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}; \tag{31}$$

$$T(p_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad T(p_2) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{32}$$

This corresponds to one of the four three-dimensional irreducible representations of $\mathrm{D}_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ (Table 2).

The general procedure in Sec. 4.1 is applied to

$$\begin{aligned} \Sigma &= \langle r, s, p_1^2, p_2^2 \rangle = \langle r, s \rangle \dot{+} \langle p_1^2, p_2^2 \rangle \\ &\simeq \mathrm{D}_6 \dot{+} (\mathbb{Z}_{n/2} \times \mathbb{Z}_{n/2}), \end{aligned} \tag{33}$$

which expresses the symmetry of the hexagon with $a = 4$ of type V (Christaller's $k = 4$ system) in Fig. 4(c). The fixed-point subspace $\mathrm{Fix}(\Sigma)$ with respect to $\tilde{T} = T$ in (31) and (32) is a one-dimensional subspace of $\mathbb{R}^3$ spanned by $(1, 1, 1)^\top$. Then, by the equivariant branching lemma, there exists a bifurcated path with the symmetry of (33).

It is mentioned that a slight extension of a pre-existing result can be utilized to obtain the concrete form of the bifurcation equations and the asymptotic form of bifurcated paths. Specifically, the irreducible representation in (31) and (32) is denoted as $T^{(3,1)}$ in [Ikeda & Murota, 2010, Chapter 16], and the flower mode solution there corresponds to the solution expressing Lösch's hexagon with $a = 4$.

Remark 4.2. For $n = 2^m$ (m is a positive integer), there are successive bifurcations associated with a hierarchy of subgroups

$$\begin{aligned} \mathrm{D}_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n) &\to \mathrm{D}_6 \dot{+} (\mathbb{Z}_{n/2} \times \mathbb{Z}_{n/2}) \\ &\to \cdots \to \mathrm{D}_6 \dot{+} (\mathbb{Z}_2 \times \mathbb{Z}_2) \\ &\to \mathrm{D}_6 \dot{+} (\mathbb{Z}_1 \times \mathbb{Z}_1) = \mathrm{D}_6, \end{aligned} \tag{34}$$

where $\to$ means the occurrence of bifurcation. These successive bifurcations produce a set of nested hexagons (see computational analysis in Sec. 5.2). The spatial period doubles successively as

$$\frac{T}{d} = 1 \to 2 \to \cdots \to \frac{n}{2} \to n, \tag{35}$$

which is called spatial period-doubling bifurcation cascade.

4.4. *Hexagons with $a = 9, 12, 16, 25$: Bifurcation point of multiplicity 6*

A hexagon with $a = 9$, 12, 16 or 25 branches from a bifurcation point of multiplicity 6. The hexagons with $a = 9 (= 3^2)$, $16 (= 4^2)$, $25 (= 5^2)$ are of type V with $\varphi = 0$, and the hexagon with $a = 12$ is of type M with $\varphi = \pi/6$.

The group $\mathrm{D}_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$, with $n \geq 3$, has six-dimensional irreducible representations. By defining

$$\begin{aligned} T^{(k,\ell,j)}(r) &= \begin{bmatrix} & & S \\ S & & \\ & S & \end{bmatrix}, \\ T^{(k,\ell,j)}(s) &= \sigma_j \begin{bmatrix} & I & \\ I & & \\ & & I \end{bmatrix}, \end{aligned} \tag{36}$$

$$\begin{aligned} T^{(k,\ell,j)}(p_1) &= \begin{bmatrix} R^k & & \\ & R^\ell & \\ & & R^{-k-\ell} \end{bmatrix}, \\ T^{(k,\ell,j)}(p_2) &= \begin{bmatrix} R^\ell & & \\ & R^{-k-\ell} & \\ & & R^k \end{bmatrix}, \end{aligned} \tag{37}$$

where $\sigma_1 = 1$, $\sigma_2 = -1$, and

$$R = \begin{bmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{38}$$

we can designate the six-dimensional irreducible representations by

$$(k, \ell, j) = (k, 0, j) \quad \text{with } 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \quad j \in \{1, 2\}; \tag{39}$$

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or

$$(k,\ell,j) = (k,k,j) \quad \text{with } 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \qquad k \neq \frac{n}{3}; j \in \{1,2\}. \tag{40}$$

The action given in (36) and (37) on six-dimensional vectors, say, $(w_1,\ldots,w_6)$, can be expressed for complex variables $(z_1,z_2,z_3) = (w_1 + \mathrm{i}w_2, w_3 + \mathrm{i}w_4, w_5 + \mathrm{i}w_6)$ as

$$r : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} \overline{z_2} \\ \overline{z_3} \\ \overline{z_1} \end{pmatrix}, \qquad s : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} \sigma_j z_2 \\ \sigma_j z_1 \\ \sigma_j z_3 \end{pmatrix}, \tag{41}$$

$$p_1 : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} \omega^k z_1 \\ \omega^\ell z_2 \\ \omega^{-k-\ell} z_3 \end{pmatrix}, \qquad p_2 : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} \omega^\ell z_1 \\ \omega^{-k-\ell} z_2 \\ \omega^k z_3 \end{pmatrix}, \tag{42}$$

where $\omega = \exp(\mathrm{i}2\pi/n)$.

Recall from Sec. 2.3 that Lösch's hexagons with $a = 9$, 12, 16, 25 are endowed with the symmetry of

$$\Sigma^{(\alpha,\beta)} = \langle r,s \rangle \dot{+} \langle p_1^\alpha p_2^\beta, p_1^{-\beta} p_2^{\alpha-\beta} \rangle, \tag{43}$$

where

$$(\alpha,\beta;n) = \begin{cases} (3,0;3m) & \text{for } a = 9, \\ (4,2;6m) & \text{for } a = 12, \\ (4,0;4m) & \text{for } a = 16, \\ (5,0;5m) & \text{for } a = 25. \end{cases} \tag{44}$$

To apply the general procedure in Sec. 4.1 to $\Sigma = \Sigma^{(\alpha,\beta)}$ in (43) we search for irreducible representations (k,ℓ,j) such that

$$\mathrm{Fix}(\Sigma^{(\alpha,\beta)}) = \{z = (z_1,z_2,z_3) \mid T^{(k,\ell,j)}(g) \cdot z = z \text{ for all } g \in \Sigma^{(\alpha,\beta)}\} \tag{45}$$

is nontrivial with $\dim \mathrm{Fix}(\Sigma^{(\alpha,\beta)}) \ge 1$. Here $T^{(k,\ell,j)}(g) \cdot z$ means the action of g given in (41) and (42), and the dependence of $\mathrm{Fix}(\Sigma^{(\alpha,\beta)})$ on (k,ℓ,j) is implicit in the notation.

Lemma 1. *For $(\alpha,\beta;n)$ in (44) we have the following.*

(i) $\mathrm{Fix}(\langle r,s \rangle) = \{(\rho,\rho,\rho) \mid \rho \in \mathbb{R}\}$ *for each (k,ℓ,j) with $j = 1$; and* $\mathrm{Fix}(\langle r,s \rangle) = \{\mathbf{0}\}$ *for each (k,ℓ,j) with $j = 2$.*
(ii) $\mathrm{Fix}(\Sigma^{(\alpha,\beta)}) = \{(\rho,\rho,\rho) \mid \rho \in \mathbb{R}\}$ *holds for*

$$(k,\ell,j;n) = \begin{cases} (m,0,1;3m) & \textit{for } a = 9, \\ \textit{none} & \textit{for } a = 12, \\ (m,0,1;4m) & \textit{for } a = 16, \\ (m,0,1;5m), \quad (2m,0,1;5m) & \textit{for } a = 25 \end{cases} \tag{46}$$

or

$$(k,\ell,j;n) = \begin{cases} \textit{none} & \textit{for } a = 9, \\ (m,m,1;6m) & \textit{for } a = 12, \\ (m,m,1;4m) & \textit{for } a = 16, \\ (m,m,1;5m), \quad (2m,2m,1;5m) & \textit{for } a = 25. \end{cases} \tag{47}$$

Proof

(i) This is immediate from (41).
(ii) The invariance of $z = (z_1,z_2,z_3) = (\rho,\rho,\rho)$ with $\rho \neq 0$ to $p_1^\alpha p_2^\beta$ is expressed as

$$\begin{aligned} k\alpha + \ell\beta &\equiv 0, \\ \ell\alpha - (k+\ell)\beta &\equiv 0, \\ -(k+\ell)\alpha + k\beta &\equiv 0 \mod n, \end{aligned} \tag{48}$$

whereas the invariance to $p_1^{-\beta} p_2^{\alpha-\beta}$ as

$$\begin{aligned} -k\beta + \ell(\alpha-\beta) &\equiv 0, \\ -\ell\beta - (k+\ell)(\alpha-\beta) &\equiv 0, \\ (k+\ell)\beta + k(\alpha-\beta) &\equiv 0 \mod n, \end{aligned}$$

which is equivalent to (48). For $(k,\ell) = (k,0)$, (48) is simplified to

$$k\alpha \equiv 0, \quad k\beta \equiv 0, \quad -k\alpha + k\beta \equiv 0 \mod n, \tag{49}$$

and the parameter values $(k,\ell) = (k,0)$ satisfying (49) in the range of (39) are enumerated

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by (46). For $(k,\ell) = (k,k)$, on the other hand, (48) is reduced to

$$\begin{aligned} k\alpha + k\beta &\equiv 0, \\ k\alpha - 2k\beta &\equiv 0, \\ -2k\alpha + k\beta &\equiv 0 \quad \bmod n, \end{aligned} \tag{50}$$

and the parameter values $(k,\ell) = (k,k)$ satisfying (50) in the range of (40) are enumerated by (47). ■

The following is the main result of this section.

Proposition 1. *Lösch's hexagons with* $a = 9, 12,$ 16, *and* 25 *arise as bifurcated solutions from bifurcation points of multiplicity* 6 *associated with the irreducible representations given in* (46) *or* (47).

Proof. For the parameter values in (46) or (47) we have $\dim \mathrm{Fix}(\Sigma^{(\alpha,\beta)}) = 1$ by Lemma 1(ii). Then the equivariant branching lemma guarantees the existence of a bifurcated solution $\mathbf{h}$ with $\Sigma(\mathbf{h}) = \Sigma^{(\alpha,\beta)}$. ■

Knowledge about the possible bifurcation points given in Proposition 1 and Lemma 1 is helpful in conducting numerical analysis.

Remark 4.3. There exist no bifurcated solutions for $a = 3$ or $a = 4$ from a bifurcation point of multiplicity 6. This follows from the fact that Eq. (48) with $(\alpha,\beta) = (2,1)$ or $(2,0)$ has no solution (k,ℓ) satisfying (39) or (40).

Remark 4.4. There exist no bifurcated solutions for $a = 7, 13, 19, 21$ (tilted hexagons) from a bifurcation point of multiplicity 6. This follows from the fact that Eq. (48) with (α,β) given later in (58) has no solution (k,ℓ) satisfying (39) or (40).

4.5. *Hexagons with* $a = 7, 13, 19, 21$*: Bifurcation point of multiplicity* 12

A hexagon with $a = 7$, 13, 19 or 21 branches from a bifurcation point of multiplicity 12. These hexagons are tilted ($\varphi \neq 0,\ \pi/6$) in contrast to the other hexagons obtained in Secs. 4.2–4.4. The emergence of such tilted hexagons is most phenomenal in the present study.

The group $\mathrm{D}_6 \dotplus (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$, with $n \geq 6$, has 12-dimensional irreducible representations. We can designate them by (k,ℓ) with

$$1 \leq \ell \leq k-1, \quad 2k + \ell \leq n-1, \tag{51}$$

where the irreducible representation (k,ℓ) is defined as

$$T^{(k,\ell)}(r) = \left[\begin{array}{ccc|ccc} & & S & & & \\ S & & & & & \\ & S & & & & \\ \hline & & & & S & \\ & & & & & S \\ & & & S & & \end{array}\right], \quad T^{(k,\ell)}(s) = \left[\begin{array}{ccc|ccc} & & & I & & \\ & & & & I & \\ & & & & & I \\ \hline I & & & & & \\ & I & & & & \\ & & I & & & \end{array}\right], \tag{52}$$

$$T^{(k,\ell)}(p_1) = \left[\begin{array}{ccc|ccc} R^k & & & & & \\ & R^\ell & & & & \\ & & R^{-k-\ell} & & & \\ \hline & & & R^k & & \\ & & & & R^\ell & \\ & & & & & R^{-k-\ell} \end{array}\right],$$

$$T^{(k,\ell)}(p_2) = \left[\begin{array}{ccc|ccc} R^\ell & & & & & \\ & R^{-k-\ell} & & & & \\ & & R^k & & & \\ \hline & & & R^{-k-\ell} & & \\ & & & & R^k & \\ & & & & & R^\ell \end{array}\right], \tag{53}$$

where

$$R = \begin{bmatrix} \cos\dfrac{2\pi}{n} & -\sin\dfrac{2\pi}{n} \\ \sin\dfrac{2\pi}{n} & \cos\dfrac{2\pi}{n} \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{54}$$

The action given in (52) and (53) on 12-dimensional vectors, say, $(w_1, \ldots, w_{12})$, can be

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expressed for complex variables $z_j = w_{2j-1} + \mathrm{i}w_{2j}$ $(j = 1, \ldots, 6)$ as

$$r : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} \mapsto \begin{pmatrix} \overline{z_3} \\ \overline{z_1} \\ \overline{z_2} \\ \overline{z_5} \\ \overline{z_6} \\ \overline{z_4} \end{pmatrix}, \quad s : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} \mapsto \begin{pmatrix} z_4 \\ z_5 \\ z_6 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix}, \tag{55}$$

$$p_1 : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} \mapsto \begin{pmatrix} \omega^k z_1 \\ \omega^\ell z_2 \\ \omega^{-k-\ell} z_3 \\ \omega^k z_4 \\ \omega^\ell z_5 \\ \omega^{-k-\ell} z_6 \end{pmatrix},$$
$$p_2 : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} \mapsto \begin{pmatrix} \omega^\ell z_1 \\ \omega^{-k-\ell} z_2 \\ \omega^k z_3 \\ \omega^{-k-\ell} z_4 \\ \omega^k z_5 \\ \omega^\ell z_6 \end{pmatrix}, \tag{56}$$

where $\omega = \exp(\mathrm{i}2\pi/n)$.

Recall from Sec. 2.3 that Lösch's hexagons with $a = 7$, 13, 19, 21 are endowed with the symmetry of

$$\Sigma_0^{(\alpha,\beta)} = \langle r \rangle \dotplus \langle p_1^\alpha p_2^\beta, p_1^{-\beta} p_2^{\alpha-\beta} \rangle, \tag{57}$$

where

$$(\alpha, \beta; n) = \begin{cases} (3, 1; 7m) & \text{for } a = 7, \\ (4, 1; 13m) & \text{for } a = 13, \\ (5, 2; 19m) & \text{for } a = 19, \\ (5, 1; 21m) & \text{for } a = 21. \end{cases} \tag{58}$$

To apply the general procedure in Sec. 4.1 to $\Sigma = \Sigma_0^{(\alpha,\beta)}$ we search for irreducible representations (k, ℓ) such that

$$\mathrm{Fix}(\Sigma_0^{(\alpha,\beta)}) = \{z = (z_1, z_2, z_3, z_4, z_5, z_6) \,|\, T^{(k,\ell)}(g) \cdot z = z \text{ for all } g \in \Sigma_0^{(\alpha,\beta)}\} \tag{59}$$

is nontrivial with $\dim \mathrm{Fix}(\Sigma_0^{(\alpha,\beta)}) \geq 1$. Here $T^{(k,\ell)}(g) \cdot z$ means the action of g given in (55) and (56), and the dependence of $\mathrm{Fix}(\Sigma_0^{(\alpha,\beta)})$ on (k, ℓ) are implicit in the notation.

Lemma 2. *For $(\alpha, \beta; n)$ in (58) we have the following.*

(i) $\mathrm{Fix}(\langle r \rangle) = \{(\rho, \rho, \rho, \rho', \rho', \rho') \,|\, \rho, \rho' \in \mathbb{R}\}$ *for each (k, ℓ).*
(ii) $\dim \mathrm{Fix}(\Sigma_0^{(\alpha,\beta)}) \leq 1$ *for each (k, ℓ).*
(iii) *If* $\dim \mathrm{Fix}(\Sigma_0^{(\alpha,\beta)}) = 1$, *then*

$$\mathrm{Fix}(\Sigma_0^{(\alpha,\beta)}) = \{(\rho, \rho, \rho, 0, 0, 0) \,|\, \rho \in \mathbb{R}\} \quad \textit{or} \tag{60}$$

$$\mathrm{Fix}(\Sigma_0^{(\alpha,\beta)}) = \{(0, 0, 0, \rho', \rho', \rho') \,|\, \rho' \in \mathbb{R}\}. \tag{61}$$

(iv) *(60) holds for*

$$(k, \ell; n) = \begin{cases} (2m, m; 7m) & \textit{for } a = 7, \\ (3m, m; 13m) & \textit{for } a = 13, \\ (3m, 2m; 19m), \\ \quad (6m, 4m; 19m) & \textit{for } a = 19, \\ (4m, m; 21m), \\ \quad (8m, 2m; 21m) & \textit{for } a = 21. \end{cases} \tag{62}$$

(v) *(61) holds for*

$$(k, \ell; n) = \begin{cases} \textit{none} & \textit{for } a = 7, \\ (5m, 2m; 13m) & \textit{for } a = 13, \\ (7m, m; 19m) & \textit{for } a = 19, \\ (6m, 3m; 21m) & \textit{for } a = 21. \end{cases} \tag{63}$$

Proof

(i) is immediate from the action of r in (55).

For the proof of (ii) to (v) we first consider the symmetry of $z = (z_1, z_2, z_3, z_4, z_5, z_6) = (\rho, \rho, \rho, 0, 0, 0)$ with $\rho \neq 0$. The invariance of such z to $p_1^\alpha p_2^\beta$ is expressed as

$$\begin{aligned} k\alpha + \ell\beta &\equiv 0, \\ \ell\alpha - (k + \ell)\beta &\equiv 0, \\ -(k + \ell)\alpha + k\beta &\equiv 0 \mod n, \end{aligned} \tag{64}$$

whereas the invariance to $p_1^{-\beta} p_2^{\alpha-\beta}$ as

$$\begin{aligned} -k\beta + \ell(\alpha - \beta) &\equiv 0, \\ -\ell\beta - (k + \ell)(\alpha - \beta) &\equiv 0, \\ (k + \ell)\beta + k(\alpha - \beta) &\equiv 0 \mod n, \end{aligned}$$

which is equivalent to (64). The parameter values (k,ℓ) satisfying (64) in the range of (51) are enumerated by (62), as can be verified easily using the relation $n=(\alpha^2-\alpha\beta+\beta^2)m$, which follows from (5) and (58). Hence we have

$$\mathrm{Fix}(\Sigma_0^{(\alpha,\beta)}) \supseteq \{(\rho,\rho,\rho,0,0,0)\,|\,\rho\in\mathbb{R}\} \tag{65}$$

for (k,ℓ) in (62).

Next we consider, in a similar manner, the symmetry of $z=(z_1,z_2,z_3,z_4,z_5,z_6)=(0,0,0,\rho',\rho',\rho')$ with $\rho'\neq 0$. The invariance of such z to $p_1^{\alpha}p_2^{\beta}$ and $p_1^{-\beta}p_2^{\alpha-\beta}$ is expressed as

$$\begin{aligned} k\alpha-(k+\ell)\beta &\equiv 0,\\ \ell\alpha+k\beta &\equiv 0,\\ -(k+\ell)\alpha+\ell\beta &\equiv 0 \mod n. \end{aligned} \tag{66}$$

The parameter values (k,ℓ) satisfying (66) in the range of (51) are enumerated by (63), and therefore

$$\mathrm{Fix}(\Sigma_0^{(\alpha,\beta)}) \supseteq \{(0,0,0,\rho',\rho',\rho')\,|\,\rho'\in\mathbb{R}\} \tag{67}$$

for (k,ℓ) in (63).

Since no (k,ℓ) is common to (62) and (63), (65) and (67) cannot be true simultaneously. Furthermore, we can see from the above argument that if $z=(\rho,\rho,\rho,\rho',\rho',\rho')\in\mathrm{Fix}(\Sigma_0^{(\alpha,\beta)})$, then we must have $\rho=0$ or $\rho'=0$. This shows the assertions (ii) and (iii). Then the assertions in (iv) and (v) follow from (65) and (67). ■

Remark 4.5. We note the relation

$$\hat{k}^2+\hat{k}\hat{\ell}+\hat{\ell}^2\equiv 0 \mod \hat{n} \tag{68}$$

as a consequence of (64), where

$$\hat{k}=\frac{k}{\gcd(k,\ell,n)},\quad \hat{\ell}=\frac{\ell}{\gcd(k,\ell,n)},$$
$$\hat{n}=\frac{n}{\gcd(k,\ell,n)},$$

in which $\gcd(k,\ell,n)$ denotes the greatest common divisor of k, ℓ, n. To see this, we first rewrite (64) as

$$\begin{aligned} \hat{k}\alpha+\hat{\ell}\beta &\equiv 0,\\ \hat{\ell}\alpha-(\hat{k}+\hat{\ell})\beta &\equiv 0,\\ -(\hat{k}+\hat{\ell})\alpha+\hat{k}\beta &\equiv 0 \mod \hat{n}, \end{aligned}$$

and then eliminate β or α to obtain

$$(\hat{k}^2+\hat{k}\hat{\ell}+\hat{\ell}^2)\alpha\equiv 0,\quad (\hat{k}^2+\hat{k}\hat{\ell}+\hat{\ell}^2)\beta\equiv 0 \mod \hat{n}.$$

This implies (68), since α or β is relatively prime to $\hat{n}$ in each case of our interest in (58).

The following is the main result of this section. Recall Fig. 4(d) for the hexagon with $a=7$.

Proposition 2. *Lösch's hexagons with $a=7,13$, 19 and 21 arise as bifurcated solutions from bifurcation points of multiplicity 12 associated with the irreducible representations given in (62) or (63).*

Proof. For the parameter values in (62) or (63) we have $\dim\mathrm{Fix}(\Sigma_0^{(\alpha,\beta)})=1$ by Lemma 2. Then the equivariant branching lemma guarantees the existence of a bifurcated solution $\mathbf{h}$ with $\Sigma(\mathbf{h})=\Sigma_0^{(\alpha,\beta)}$. ■

Detailed analysis of the bifurcation equations is carried out in Appendix B. It shows, for example, that the ρ versus τ curve at the bifurcation point at $(\rho,\tau)=(0,0)$ is given asymptotically as $A\tau+B\rho=0$. Knowledge about the bifurcated solutions obtained through this analysis, as well as about the bifurcation points stated in Lemma 2, is helpful in conducting numerical analysis.

Remark 4.6. Bifurcated solutions representing hexagons of type V from a bifurcation point of multiplicity 12 are considered here. Such solutions are characterized by the symmetry $\langle r,s\rangle\dot{+}\langle p_1^{\alpha},p_2^{\alpha}\rangle$ with $\alpha\geq 2$, which can be denoted as $\Sigma^{(\alpha,0)}$, i.e. $\Sigma^{(\alpha,\beta)}$ with $\beta=0$, in the notation of (43). Then we have $a=\alpha^2$ in (5). First, by (55) we have

$$\mathrm{Fix}(\langle r,s\rangle)=\{(\rho,\rho,\rho,\rho,\rho,\rho)\,|\,\rho\in\mathbb{R}\} \tag{69}$$

for each (k,ℓ). We have $\dim\mathrm{Fix}(\Sigma^{(\alpha,0)})=1$ if (k,ℓ) satisfies (64) and (66) for $\beta=0$; otherwise $\dim\mathrm{Fix}(\Sigma^{(\alpha,0)})=0$. This condition for (k,ℓ) reduces to $k\alpha\equiv\ell\alpha\equiv 0 \mod n$, where (k,ℓ) must lie in the range of (51). Then we must have $n=\alpha m$ for some integer m, and (k,ℓ) is given as $(k,\ell)=(pm,qm)$ with

$$1\leq q\leq p-1,\quad 2p+q\leq\alpha-1,\quad p,q\in\mathbb{Z}.$$

Such (k,ℓ) does not exist for $\alpha\leq 5$, showing that there exist no bifurcated solutions from a bifurcation point of multiplicity 12 that represent Lösch's hexagons with $a=4,9,16,25$ associated respectively with $\alpha=2,3,4,5$ (Table 1). For $\alpha\geq 6$, on the other hand, the following parameter values satisfy the above-mentioned condition.

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a	(α, β)	n	(k, ℓ)
36	$(6, 0)$	$6m$	$(2m, m)$
49	$(7, 0)$	$7m$	$(2m, m)$
64	$(8, 0)$	$8m$	$(2m, m), (3m, m)$
81	$(9, 0)$	$9m$	$(2m, m), (3m, m), (3m, 2m)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

These parameter values generically give rise to bifurcated solutions representing hexagons of type V. It is noted that when $m = 1$, we have $\alpha = n$, and hence the symmetry $\langle r, s\rangle \dot{+} \langle p_1^\alpha, p_2^\alpha\rangle$ reduces to $\langle r, s\rangle = \mathrm{D}_6$ (see Remark 4.8).

Remark 4.7. Bifurcated solutions representing hexagons of type M from a bifurcation point of multiplicity 12 are considered here. Such solutions are characterized by the symmetry $\langle r, s\rangle \dot{+} \langle p_1^{2\beta} p_2^{\beta}, p_1^{-\beta} p_2^{\beta}\rangle$, which can be denoted as $\Sigma^{(2\beta,\beta)}$, i.e. $\Sigma^{(\alpha,\beta)}$ with $\alpha = 2\beta$, in the notation of (43). Then we have $a = 3\beta^2$ in (5). The expression (69) for the subspace $\mathrm{Fix}(\langle r, s\rangle)$ is again valid for each (k, ℓ). We have $\dim \mathrm{Fix}(\Sigma^{(2\beta,\beta)}) = 1$ if (k, ℓ) satisfies (64) and (66) for $(\alpha, \beta) = (2\beta, \beta)$; otherwise $\dim \mathrm{Fix}(\Sigma^{(2\beta,\beta)}) = 0$. This condition is equivalent to

$$(2k + \ell)\beta \equiv (k + 2\ell)\beta \equiv (k - \ell)\beta \equiv 0 \mod n,$$

where (k, ℓ) must lie in the range of (51). Then we must have $n = 3\beta m$ for some integer m, and (k, ℓ) is given as $(k, \ell) = (pm, qm)$ with

$$1 \le q \le p - 1, \quad p - q \equiv 0 \mod 3,$$
$$2p + q \le 3\beta - 1, \quad p, q \in \mathbb{Z}.$$

Such (k, ℓ) does not exist for $\beta \le 3$, showing that there exist no bifurcated solutions from a bifurcation point of multiplicity 12 that represent Lösch's hexagons with $a = 3, 12$ associated respectively with $\beta = 1, 2$ (Table 1). For $\beta \ge 4$, on the other hand, the following parameter values satisfy the above-mentioned condition.

a	(α, β)	n	(k, ℓ)
48	$(8, 4)$	$12m$	$(4m, m)$
75	$(10, 5)$	$15m$	$(4m, m), (5m, 2m)$
108	$(12, 6)$	$18m$	$(4m, m), (5m, 2m), (6m, 3m), (7m, m)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

These parameter values generically give rise to bifurcated solutions representing hexagons of type M.

Remark 4.8. Bifurcated solutions with D_6-symmetry are considered here. Take $z = (\rho, \rho, \rho, \rho, \rho, \rho)$ with $\rho \ne 0$. Since $z \in \mathrm{Fix}(\langle r, s\rangle)$ by (69), we have $\Sigma(z) \supseteq \langle r, s\rangle$. We often have $\Sigma(z) = \langle r, s\rangle$, since, except for some special values of (k, ℓ) such as those listed in Remarks 4.6 and 4.7, there exists no nontrivial (α, β) that satisfies (64) and (66). Note in this connection that we must have

$$(k - \ell)\alpha \equiv (2k + \ell)\alpha \equiv (2\ell + k)\alpha \equiv 0 \mod n,$$
$$(k - \ell)\beta \equiv (2k + \ell)\beta \equiv (2\ell + k)\beta \equiv 0 \mod n$$

as a consequence of (64) and (66). For $(k, \ell; n) = (k, k - 1; n)$ with $2 \le k \le n/3$, for example, we must have $\alpha \equiv \beta \equiv 0 \mod n$, and hence $\Sigma(z) = \langle r, s\rangle$. For the parameter values of (k, ℓ) for which $\Sigma(z) = \langle r, s\rangle$ holds, the subgroup $\mathrm{D}_6 = \langle r, s\rangle$ is an axial subgroup and, by the equivariant branching lemma, there exist bifurcated solutions with D_6-symmetry (lacking translational symmetry). It is noted that the normalized spatial period is given as $T/d = \sqrt{a} = n$ with $(\alpha, \beta) = (n, 0)$ or $(0, n)$ in (4) and (5). Figure 5 illustrates the pattern of such solution for $(k, \ell; n) = (2, 1; 6)$. D_6-symmetric bifurcated solutions correspond to "super hexagons" investigated for the group $\mathrm{D}_6 \dot{+} \mathrm{T}^2$ in [Kirchgässner, 1979; Dionne *et al.*, 1997].

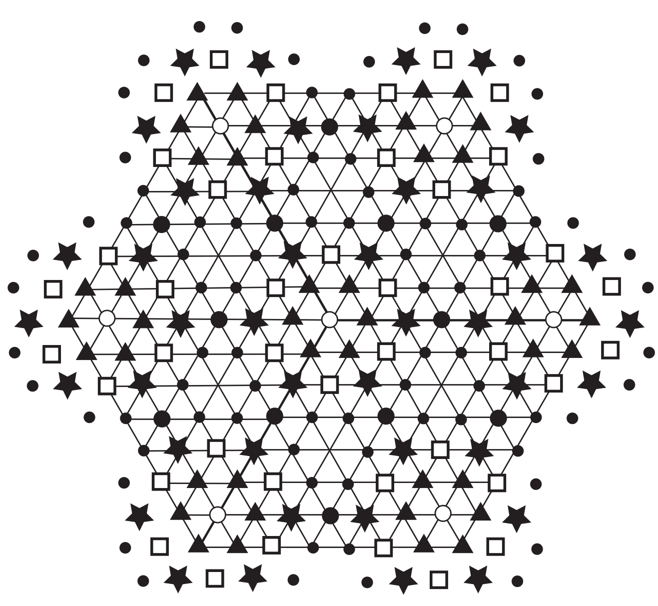

Fig. 5. Pattern with D_6-symmetry ($n = 6$).

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5. Computationally Obtained Hexagonal Distributions

In this section, we examine spatial agglomeration patterns of the population of skilled workers among a system of places spread uniformly on a two-dimensional domain. Computational bifurcation analysis is conducted to obtain bifurcated solutions from the uniformly distributed state of population of the skilled workers for a system of $n \times n$ places on the hexagonal lattice for $n = 9, 16$, and 19 to observe several Lösch's ten smallest hexagons. We employ the following parameter values:

- The length d of the road connecting neighboring places is $d = 1/n$.
- The constant expenditure share μ on industrial varieties is $\mu = 0.4$.
- The constant elasticity σ of substitution between any two varieties is $\sigma = 5.0$.
- The inverse θ of variance of the idiosyncratic tastes is $\theta = 1000$.
- The total number H of skilled workers is $H = 1$.

5.1. *Hexagons with $a = 3$ and $a = 9$ for 9×9 places*

For the 9×9 places with $\mathrm{D}_6 \dotplus (\mathbb{Z}_9 \times \mathbb{Z}_9)$-symmetry, we conducted the computational bifurcation analysis to obtain the maximum population $h_{\max}$ versus the transport parameter τ curves in Fig. 6(a). Although several bifurcation points are found on the trivial solution OABC with uniform population, we specifically examine the bifurcation points A and B of multiplicity $M = 2$, from which a hexagonal distribution with $a = 3$ emanates, and the bifurcation point C of multiplicity $M = 6$, from which a hexagonal distribution with $a = 9$ emanates. Among many bifurcation points of multiplicity $M = 6$, we have chosen the bifurcation point C with the kernel symmetry $\langle p_1^3, p_2^3 \rangle$.

On the bifurcated path ADB that branches from the bifurcation points A and B of multiplicity $M = 2$, we encounter Lösch's smallest hexagon with $a = 3$ that has $\langle r, s, p_1^2 p_2, p_1^{-1} p_2 \rangle$-symmetry and the spatial period $T/d = \sqrt{a} = \sqrt{3}$ (Sec. 4.2).

On the bifurcated path DEC that branches from the bifurcation point C of multiplicity $M = 6$, we encounter Lösch's fourth smallest hexagon with $a = 9$ that has $\langle r, s, p_1^3, p_2^3 \rangle$-symmetry and has the spatial period $T/d = \sqrt{a} = 3$ (Sec. 4.4).

At the bifurcation point D of multiplicity 2 on the primary bifurcated path ADB, we encounter a secondary bifurcation. This is the spatial period $\sqrt{3}$-times cascade (Remark 4.1), in which the spatial period T is extended $\sqrt{3}$-times repeatedly as

$$
\begin{array}{rcccccc}
\dfrac{T}{d}: & 1 & \rightarrow & \sqrt{3} & \rightarrow & 3 \\
(\mathbf{t}_1, \mathbf{t}_2): & (\boldsymbol{\ell}_1, \boldsymbol{\ell}_2) & \rightarrow & (2\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2, -\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2) & \rightarrow & (3\boldsymbol{\ell}_1, 3\boldsymbol{\ell}_2) \\
\text{group}: & \mathrm{D}_6 \dotplus (\mathbb{Z}_9 \times \mathbb{Z}_9) & \rightarrow & \mathrm{D}_6 \dotplus (\mathbb{Z}_9 \times \mathbb{Z}_3) & \rightarrow & \mathrm{D}_6 \dotplus (\mathbb{Z}_3 \times \mathbb{Z}_3) \\
\text{path}: & \text{OABC} & \rightarrow & \text{ADB} & \rightarrow & \text{DEC}
\end{array}
$$

5.2. *Hexagons with $a = 4$ and $a = 16$ for 16×16 places*

For the 16×16 places with $\mathrm{D}_6 \dotplus (\mathbb{Z}_{16} \times \mathbb{Z}_{16})$-symmetry, Fig. 6(b) shows the maximum population $h_{\max}$ versus the transport parameter τ curves. Several bifurcation points are found on the trivial solution OABC with uniform population. We specifically examine the bifurcation points A and B of multiplicity $M = 3$, from which a hexagonal distribution with $a = 4$ emanates, and the bifurcation point C of multiplicity $M = 6$, from which a hexagonal distribution with $a = 16$ emanates. Among many bifurcation points of multiplicity $M = 6$, we chose the bifurcation point C with the kernel symmetry $\langle p_1^4, p_2^4 \rangle$.

On the bifurcated path ADB that branches from the bifurcation points A and B of multiplicity $M = 3$, we encounter Lösch's smallest hexagon with $a = 4$ that has $\langle r, s, p_1^2, p_2^2 \rangle$-symmetry and the spatial period $T/d = \sqrt{a} = \sqrt{4}$ (Sec. 4.3).

On the bifurcated path DEC that branches from the bifurcation point C of multiplicity $M = 6$, we encounter Lösch's six–seventh smallest hexagon with $a = 16$ that has $\langle r, s, p_1^4, p_2^4 \rangle$-symmetry and has the spatial period $T/d = \sqrt{a} = \sqrt{16}$ (Sec. 4.4).

At the bifurcation point D of multiplicity 3 on the primary bifurcated path ADB, we encounter a secondary bifurcation. This is the spatial period-doubling cascade, in which the spatial period T is doubled repeatedly as

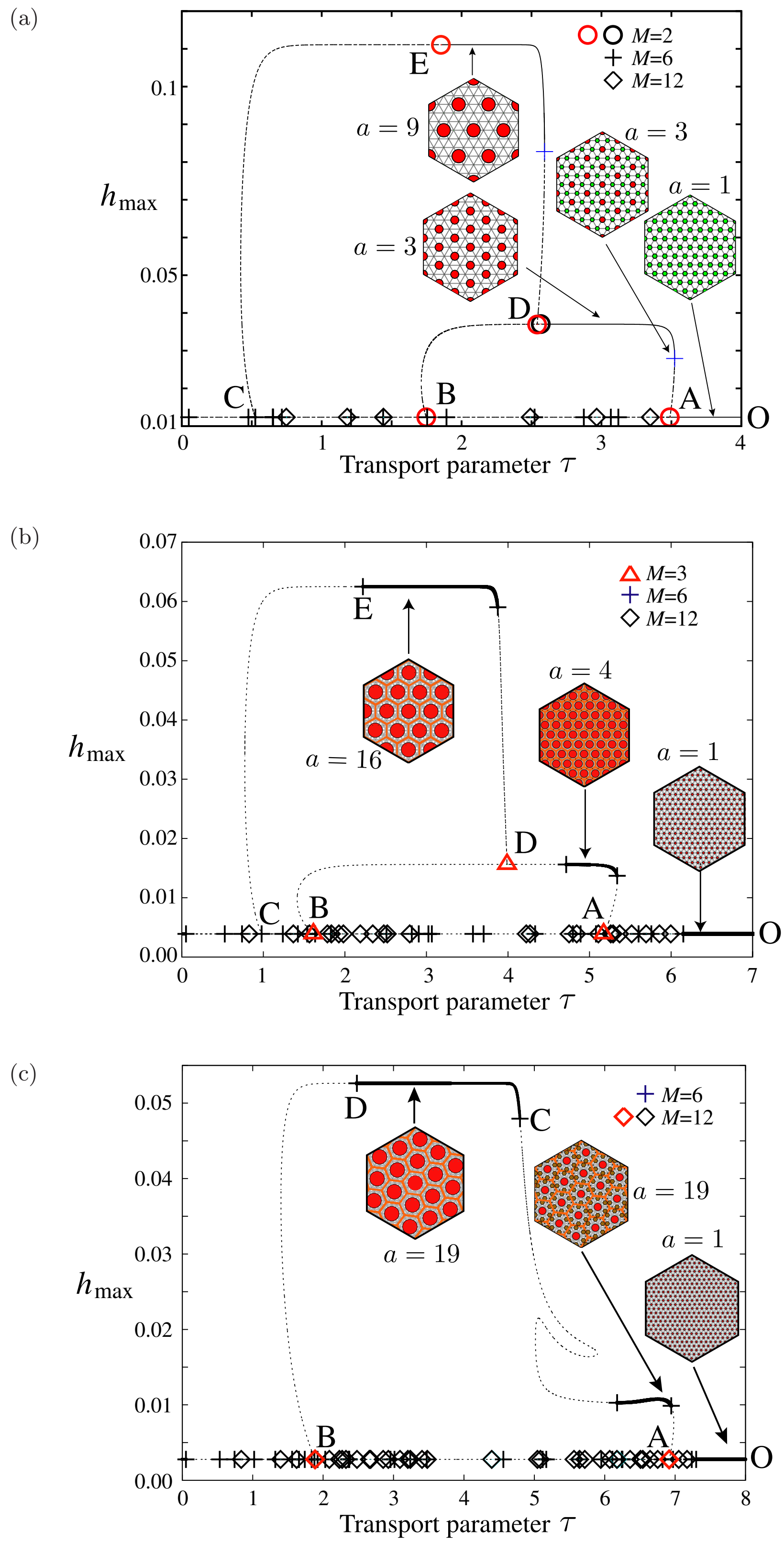

Fig. 6. Maximum population $h_{\max}$ versus the transport parameter τ curves. (The solid curve is stable and the dashed curve is unstable. The hexagonal window is cut from the infinite domain that is obtained by repeating the $n \times n$ hexagonal lattice spatially; and the area of each circle is proportional to the population of the skilled workers at that place.) (a) Hexagons with $a = 3$ and $a = 9$ for 9×9 places, (b) Hexagons with $a = 4$ and $a = 16$ for 16×16 places, and (c) Hexagon with $a = 19$ for 19×19 places.

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$$\begin{array}{lccccc} \dfrac{T}{d}: & 1 & \rightarrow & 2 & \rightarrow & 2^2 \\ (\mathbf{t}_1, \mathbf{t}_2): & (\boldsymbol{\ell}_1, \boldsymbol{\ell}_2) & \rightarrow & (2\boldsymbol{\ell}_1, 2\boldsymbol{\ell}_2) & \rightarrow & (2^2\boldsymbol{\ell}_1, 2^2\boldsymbol{\ell}_2) \\ \text{group}: & \mathrm{D}_6 \dot{+} (\mathbb{Z}_{16} \times \mathbb{Z}_{16}) & \rightarrow & \mathrm{D}_6 \dot{+} (\mathbb{Z}_8 \times \mathbb{Z}_8) & \rightarrow & \mathrm{D}_6 \dot{+} (\mathbb{Z}_4 \times \mathbb{Z}_4) \\ \text{path}: & \text{OABC} & \rightarrow & \text{ADB} & \rightarrow & \text{DEC} \end{array}$$

This hierarchy is in agreement with the theoretically predicted hierarchy (34) and (35) for $n = 16$.

5.3. *Hexagons with* $a = 19$ *for* 19×19 *places*

For the 19×19 places with $\mathrm{D}_6 \dot{+} (\mathbb{Z}_{19} \times \mathbb{Z}_{19})$-symmetry, Fig. 6(c) shows the maximum population $h_{\max}$ versus the transport parameter τ curves. Several bifurcation points are found on the trivial solution OAB with uniform population. We specifically examine the bifurcation points A and B of multiplicity $M = 12$ from which hexagonal distributions of interest emanate. On the bifurcated path ACDB that branches from these two bifurcation points A and B we encounter Lösch's eighth smallest hexagon with $a = 19$ that has $\langle r, p_1^5 p_2^2, p_1^{-2} p_2^3 \rangle$-symmetry and the spatial period $T/d = \sqrt{a} = \sqrt{19}$ (Sec. 4.5).

6. Conclusion

For a two-dimensional system modeled by a core-periphery model in new economic geography, self-organization of hexagonal population distributions for Lösch's ten smallest hexagons in central place theory is predicted by equivariant bifurcation analysis, and its existence is verified by computational bifurcation analysis. The equivariant bifurcation analysis has displayed its usefulness to predict possible bifurcating agglomeration patterns among a system of places in two dimensions, often associated with successive elongation of spatial periods.

Information about symmetries of bifurcated solutions offered by the equivariant bifurcation analysis is important in the computational analysis for choosing a bifurcation point that produces hexagonal distributions of interest. In particular, it is to be emphasized that tilted hexagons (super hexagons) that are directed towards different directions than the original hexagonal lattice do branch from bifurcation points of multiplicity 12.

The inherent capability of the core-periphery model to express those systems, provided with pertinent spatial platforms, is demonstrated. Major results of this paper, in principle, are applicable to other core-periphery models, and its application to other core-periphery models is a topic in the future.

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Appendix A

Core-Periphery Model

Details of the core-periphery model in Sec. 3 are presented. After presenting basic assumptions, we describe the short-run equilibrium and define the long-run equilibrium and its stability.

A.1. *Basic assumptions*

Preferences U over the M- and A-sector goods are identical across individuals, where M signifies manufacture and A stands for agriculture. The utility of an individual in place i is[3]

$$U(C_i^{\mathrm{M}}, C_i^{\mathrm{A}}) = \mu \ln C_i^{\mathrm{M}} + (1-\mu)\ln C_i^{\mathrm{A}}, \quad (0 < \mu < 1), \tag{A.1}$$

where μ is the constant expenditure share on industrial varieties, C_i^{A} is the consumption of the A-sector product in place i, and C_i^{M} is the

[3]We take logarithms of the Forslid and Ottaviano [2003] type (i.e. Cobb–Douglas-type) utility function to facilitate the analysis. This transformation has no influence on the properties of the model.

manufacturing aggregate in place i and is defined as

$$C_i^{\mathrm{M}} \equiv \left(\sum_j \int_0^{n_j} q_{ji}(k)^{(\sigma-1)/\sigma} dk \right)^{\sigma/(\sigma-1)},$$

where $q_{ji}(k)$ is the consumption in place i of a variety $k \in [0, n_j]$ produced in place j, n_j is the continuum range of varieties produced in place j, often called the number of available varieties, and $\sigma > 1$ is the constant elasticity of substitution between any two varieties. The budget constraint is given as

$$p_i^{\mathrm{A}} C_i^{\mathrm{A}} + \sum_j \int_0^{n_j} p_{ji}(k) q_{ji}(k) dk = Y_i, \quad \text{(A.2)}$$

where p_i^{A} is the price of A-sector goods in place i, $p_{ji}(k)$ is the price of a variety k in place i produced in place j and Y_i is the income of an individual in place i. The incomes (wages) of the skilled worker and the unskilled worker are represented, respectively, by w_i and w_i^{L}. We denote by K the number of places, and therefore i and j run through 1 to K.

An individual in place i maximizes (A.1) subject to (A.2). This yields the following demand functions:

$$C_i^{\mathrm{A}} = (1-\mu)\frac{Y_i}{p_i^{\mathrm{A}}}, \quad C_i^{\mathrm{M}} = \mu \frac{Y_i}{\rho_i},$$
$$q_{ji}(k) = \mu \frac{p_i^{\mathrm{A}} \rho_i^{\sigma-1} Y_i}{p_{ji}(k)^{\sigma}}, \quad \text{(A.3)}$$

where ρ_i denotes the price index of the differentiated product in place i, which is

$$\rho_i = \left(\sum_j \int_0^{n_j} p_{ji}(k)^{1-\sigma} dk \right)^{1/(1-\sigma)}. \quad \text{(A.4)}$$

Since the total income and population in place i are $w_i h_i + w_i^{\mathrm{L}}$ and $h_i + 1$, respectively, we have the total demand $Q_{ji}(k)$ in place i for a variety k produced in place j:

$$Q_{ji}(k) = \mu \frac{p_i^{\mathrm{A}} \rho_i^{\sigma-1}}{p_{ji}(k)^{\sigma}} (w_i h_i + w_i^{\mathrm{L}}). \quad \text{(A.5)}$$

The A-sector is perfectly competitive and produces homogeneous goods under constant returns to scale technology, which requires one unit of unskilled labor in order to produce one unit of output. For simplicity, we assume that the A-sector goods are transported freely between places and that they are chosen as the numèraire. These assumptions mean that, in equilibrium, the wage of an unskilled worker w_i^{L} is equal to the price of A-sector goods in all places (i.e. $p_i^{\mathrm{A}} = w_i^{\mathrm{L}} = 1$ for each $i = 1, \ldots, K$).

The M-sector output is produced under increasing returns to scale technology and Dixit–Stiglitz monopolistic competition. A firm incurs a fixed input requirement of α units of skilled labor and a marginal input requirement of β units of unskilled labor. Given the fixed input requirement α, the skilled labor market clearing implies that, in equilibrium, the number of firms in place i is determined by $n_i = h_i/\alpha$. An M-sector firm located in place i chooses $(p_{ij}(k) \,|\, j = 1, \ldots, K)$ that maximizes its profit

$$\Pi_i(k) = \sum_j p_{ij}(k) Q_{ij}(k) - (\alpha w_i + \beta x_i(k)),$$

where $x_i(k)$ is the total supply. The transportation costs for M-sector goods are assumed to take the iceberg form.[4] That is, for each unit of M-sector goods transported from place i to place $j \neq i$, only a fraction $1/\phi_{ij} < 1$ arrives. Consequently, the total supply $x_i(k)$ is given as

$$x_i(k) = \sum_j \phi_{ij} Q_{ij}(k). \quad \text{(A.6)}$$

To put it concretely, we define the transport cost ϕ_{ij} between the two places i and j as

$$\phi_{ij} = \exp(\tau D_{ij}), \quad \text{(A.7)}$$

where τ is the transport parameter and D_{ij} represents the shortest distance between places i and j.

Since we have a continuum of firms, each firm is negligible in the sense that its action has no impact on the market (i.e. the price indices). Therefore, the first-order condition for profit maximization gives

$$p_{ij}(k) = \frac{\sigma\beta}{\sigma-1}\phi_{ij}. \quad \text{(A.8)}$$

This expression implies that the price of the M-sector product does not depend on variety k, so that $Q_{ij}(k)$ and $x_i(k)$ do not depend on k. Therefore, we describe these variables without the argument k. Substituting (A.8) into (A.4), we have the

[4]This is a standard term in economics; see, for example [Fujita *et al.*, 1999].

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price index

$$\rho_i = \frac{\sigma\beta}{\sigma-1}\left(\frac{1}{\alpha}\sum_j h_j d_{ji}\right)^{1/(1-\sigma)}, \qquad \text{(A.9)}$$

where $d_{ji} = \phi_{ji}^{1-\sigma}$ is a spatial discounting factor between places j and i; from (A.5) and (A.9), d_{ji} is obtained as $(p_{ji}Q_{ji})/(p_{ii}Q_{ii})$, which means that d_{ji} is the ratio of total expenditure in place i for each M-sector product produced in place j to the expenditure for a domestic product.

A.2. *Short-run equilibrium*

In the short run, skilled workers are immobile between places, i.e. their spatial distribution ($\mathbf{h} = (h_i) \in \mathbb{R}^K$) is assumed to be given. The short-run equilibrium conditions consist of the M-sector goods market clearing condition and the zero-profit condition because of the free entry and exit of firms. The former condition can be written as (A.6). The latter condition requires that the operating profit of a firm is absorbed entirely by the wage bill of its skilled workers:

$$w_i(\mathbf{h},\tau) = \frac{1}{\alpha}\left\{\sum_j p_{ij}Q_{ij}(\mathbf{h},\tau) - \beta x_i(\mathbf{h},\tau)\right\}. \qquad \text{(A.10)}$$

Substituting (A.5), (A.6), (A.8) and (A.9) into (A.10), we have the short-run equilibrium wage:

$$w_i(\mathbf{h},\tau) = \frac{\mu}{\sigma}\sum_j \frac{d_{ij}}{\Delta_j(\mathbf{h},\tau)}(w_j(\mathbf{h},\tau)h_j + 1), \qquad \text{(A.11)}$$

where $\Delta_j(\mathbf{h},\tau) \equiv \sum_k d_{kj}h_k$ denotes the market size of the M-sector in place j. Consequently, $d_{ij}/\Delta_j(\mathbf{h},\tau)$ defines the market share in place j of each M-sector product produced in place i.

The indirect utility $v_i(\mathbf{h},\tau)$ is obtained by substituting (A.3), (A.9), and (A.11) into (A.1):[5]

$$v_i(\mathbf{h},\tau) = S_i(\mathbf{h},\tau) + \ln[w_i(\mathbf{h},\tau)], \qquad \text{(A.12)}$$

where

$$S_i(\mathbf{h},\tau) \equiv \mu(\sigma-1)^{-1}\ln\Delta_i(\mathbf{h},\tau).$$

For convenience in conducting the following analysis, we express the indirect utility function $\mathbf{v}(\mathbf{h},\tau)$ in vector form, using the spatial discounting matrix $\mathbf{D} = (d_{ij})$, as

$$\mathbf{v}(\mathbf{h},\tau) = \mathbf{S}(\mathbf{h},\tau) + \ln[\mathbf{w}(\mathbf{h},\tau)], \qquad \text{(A.13)}$$

$$\mathbf{w}(\mathbf{h},\tau) = \frac{\mu}{\sigma}[\mathbf{I} - \mathbf{W}(\mathbf{h},\tau)]^{-1}\mathbf{w}^{(\mathrm{L})}(\mathbf{h},\tau), \qquad \text{(A.14)}$$

where

$$\mathbf{S}(\mathbf{h},\tau) \equiv [S_1(\mathbf{h},\tau),\ldots,S_K(\mathbf{h},\tau)]^\top,$$

$$\ln[\mathbf{w}] \equiv [\ln w_1, \ln w_2, \ldots, \ln w_K]^\top,$$

$\mathbf{I}$ is a unit matrix, and $\mathbf{W}(\mathbf{h},\tau)$, $\mathbf{w}^{(\mathrm{H})}$, $\mathbf{w}^{(\mathrm{L})}$ and $\mathbf{M}$ are defined as

$$\mathbf{W} \equiv \frac{\mu}{\sigma}\mathbf{M}\,\mathrm{diag}[\mathbf{h}], \quad \mathbf{w}^{(\mathrm{H})} \equiv \mathbf{M}\mathbf{h}, \quad \mathbf{w}^{(\mathrm{L})} \equiv \mathbf{M}\mathbf{1}, \qquad \text{(A.15a)}$$

$$\mathbf{M} \equiv \mathbf{D}\boldsymbol{\Delta}^{-1}, \quad \boldsymbol{\Delta} \equiv \mathrm{diag}[\mathbf{D}^\top\mathbf{h}], \quad \mathbf{1} \equiv [1,\ldots,1]^\top. \qquad \text{(A.15b)}$$

A.3. *Adjustment process, long-run equilibrium and stability*

In the long run, the skilled workers are interregionally mobile. They are assumed to be heterogeneous in their preferences for location choice. That is, the indirect utility for an individual s in place i is expressed as

$$v_i^{(s)}(\mathbf{h},\tau) = v_i(\mathbf{h},\tau) + \epsilon_i^{(s)}.$$

In this equation, $\epsilon_i^{(s)}$, which is distributed continuously across individuals, denotes the utility representing the idiosyncratic taste for residential location.

We present the dynamics of the migration of the skilled workers to define the long-run equilibrium and its stability with respect to small perturbations (i.e. local stability). We assume that at each time period t, the opportunity for skilled workers to migrate emerges according to an independent Poisson process with arrival rate λ. That is, for each time interval $[t, t+dt)$, a fraction λdt of skilled workers have the opportunity to migrate. Given an opportunity at time t, each worker chooses the place that provides the highest indirect utility $v_i^{(s)}(\mathbf{h},\tau)$,

[5]We ignore the constant terms, which have no influence on the results below.

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which depends on the current distribution $\mathbf{h} = \mathbf{h}(t)$. The fraction of skilled workers who choose place i under distribution $\mathbf{h}$ is $P_i(\mathbf{v}(\mathbf{h}), \tau)$, where

$$P_i(\mathbf{v}, \tau) = \Pr[v_i^{(s)} > v_j^{(s)}, \forall j \neq i].$$

Therefore, we have

$$h_i(t + dt) = (1 - \lambda dt)h_i(t) + \lambda dt H P_i(\mathbf{v}(\mathbf{h}(t)), \tau).$$

By normalizing the unit of time so that $\lambda = 1$, we obtain the following adjustment process:

$$\dot{\mathbf{h}}(t) = \mathbf{F}(\mathbf{h}(t), \tau) \equiv H\mathbf{P}(\mathbf{v}(\mathbf{h}(t)), \tau) - \mathbf{h}(t), \tag{A.16}$$

where $\dot{\mathbf{h}}(t)$ denotes the time derivative of $\mathbf{h}(t)$, and $\mathbf{P}(\mathbf{v}(\mathbf{h}), \tau) = (P_i(\mathbf{v}(\mathbf{h}), \tau))$. For the specific functional form of $P_i(\mathbf{v}, \tau)$, we use the logit choice function:

$$P_i(\mathbf{v}, \tau) \equiv \frac{\exp[\theta v_i]}{\sum_j \exp[\theta v_j]}, \tag{A.17}$$

where $\theta \in (0, \infty)$ is the parameter denoting the inverse of variance of the idiosyncratic tastes. This implies the assumption that the distributions of $(\epsilon_i^{(s)})$'s are Gumbel distributions, which are identical and independent across places [McFadden, 1974]. The adjustment process described by (A.16) and (A.17) is the logit dynamics, which has been studied in evolutionary game theory [Fudenberg & Levine, 1998; Hofbauer & Sandholm, 2002; Sandholm, 2010].

Next, we define the long-run equilibrium and its stability. The long-run equilibrium is a stationary point of the adjustment process of (A.16).

Definition A.1. The long-run equilibrium is defined as the distribution $\mathbf{h}^*$ that satisfies

$$\mathbf{F}(\mathbf{h}^*, \tau) \equiv H\mathbf{P}(\mathbf{v}(\mathbf{h}^*), \tau) - \mathbf{h}^* = \mathbf{0}. \tag{A.18}$$

The heterogeneous worker case includes the conventional homogeneous worker case. Indeed, when $\theta \to \infty$, the condition given in (A.18) reduces to that for the homogeneous worker case:

$$\begin{cases} V^* - v_i(\mathbf{h}^*, \tau) = 0 & \text{if } h_i^* > 0, \\ V^* - v_i(\mathbf{h}^*, \tau) \geq 0 & \text{if } h_i^* = 0, \end{cases}$$

where V^* denotes the equilibrium utility.

We restrict our concern to the neighborhood of $\mathbf{h}^*$, and define the stability of $\mathbf{h}^*$ in the sense of asymptotic stability, the precise definition of which is the following.

Definition A.2. A long-run equilibrium $\mathbf{h}^*$ is *asymptotically stable* if, for any $\epsilon > 0$, there is a neighborhood $N(\mathbf{h}^*)$ of $\mathbf{h}^*$ such that, for every $\mathbf{h}_0 \in N(\mathbf{h}^*)$, the solution $\mathbf{h}(t)$ of (A.16) with an initial value $\mathbf{h}(0) \equiv \mathbf{h}_0$ satisfies $||\mathbf{h}(t) - \mathbf{h}^*|| < \epsilon$ for any time $t \geq 0$, and $\lim_{t\to\infty} \mathbf{h}(t) = \mathbf{h}^*$. It is *unstable* if equilibrium $\mathbf{h}^*$ is not asymptotically stable.

In dynamic system theory, $\mathbf{h}^*$ is asymptotically stable if all the eigenvalues of the Jacobian matrix $\nabla\mathbf{F}(\mathbf{h}, \tau) \equiv (\partial F_i(\mathbf{h}, \tau)/\partial h_j)$ of the adjustment process of (A.16) have negative real parts; otherwise $\mathbf{h}^*$ is unstable [Hirsch & Smale, 1974]. Therefore, the asymptotic stability can be assessed by examining the following Jacobian matrix:

$$\nabla\mathbf{F}(\mathbf{h}, \tau) = H\mathbf{J}(\mathbf{v}(\mathbf{h}), \tau)\nabla\mathbf{v}(\mathbf{h}, \tau) - \mathbf{I}, \tag{A.19}$$

where $\mathbf{J}(\mathbf{v}, \tau)$ and $\nabla\mathbf{v}(\mathbf{h}, \tau)$ are K-by-K matrices, the (i, j) entries of which are, respectively, $\partial P_i(\mathbf{v}, \tau)/\partial v_j$ and $\partial v_i(\mathbf{h}, \tau)/\partial h_j$. For the logit choice function of (A.17), it is easily verified that the former Jacobian matrix $\mathbf{J}(\mathbf{v}, \tau)$ is expressed as

$$\mathbf{J}(\mathbf{v}, \tau) = \theta\{\text{diag}[\mathbf{P}(\mathbf{v}, \tau)] - \mathbf{P}(\mathbf{v}, \tau)\mathbf{P}(\mathbf{v}, \tau)^\top\}. \tag{A.20}$$

The latter Jacobian matrix $\nabla\mathbf{v}(\mathbf{h}, \tau)$ is given as

$$\nabla\mathbf{v}(\mathbf{h}, \tau) = \nabla\mathbf{S}(\mathbf{h}, \tau) + \text{diag}[\mathbf{w}(\mathbf{h}, \tau)]^{-1}\nabla\mathbf{w}(\mathbf{h}, \tau), \tag{A.21}$$

$$\nabla\mathbf{w}(\mathbf{h}, \tau) = \frac{\mu}{\sigma}[\mathbf{I} - \mathbf{W}(\mathbf{h}, \tau)]^{-1} \times \{\nabla\hat{\mathbf{w}}^{(\mathrm{H})}(\mathbf{h}, \tau) + \nabla\mathbf{w}^{(\mathrm{L})}(\mathbf{h}, \tau)\}, \tag{A.22}$$

where the matrices $\nabla\mathbf{S}(\mathbf{h}, \tau)$, $\nabla\hat{\mathbf{w}}^{(\mathrm{H})}(\mathbf{h}, \tau)$, $\nabla\mathbf{w}^{(\mathrm{H})}(\mathbf{h}, \tau)$ and $\nabla\mathbf{w}^{(\mathrm{L})}(\mathbf{h}, \tau)$ are obtained as

$$\nabla\mathbf{S}(\mathbf{h}, \tau) = \mu(\sigma - 1)^{-1}\mathbf{M}^\top, \tag{A.23}$$

$$\nabla\hat{\mathbf{w}}^{(\mathrm{H})}(\mathbf{h}, \tau) = \mathbf{M}\,\text{diag}[\mathbf{w}(\mathbf{h}, \tau)] - \mathbf{M}\,\text{diag}[\mathbf{w}(\mathbf{h}, \tau)]\text{diag}[\mathbf{h}]\mathbf{M}^\top, \tag{A.24}$$

$$\nabla\mathbf{w}^{(\mathrm{H})}(\mathbf{h}, \tau) = \mathbf{M} - \mathbf{M}\,\text{diag}[\mathbf{h}]\mathbf{M}^\top, \tag{A.25}$$

$$\nabla\mathbf{w}^{(\mathrm{L})}(\mathbf{h}, \tau) = -\mathbf{M}\mathbf{M}^\top. \tag{A.26}$$

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Appendix B

Bifurcated Solutions at Bifurcation Point of Multiplicity 12

We consider a group-theoretic bifurcation point of multiplicity 12. To investigate Lösch's hexagons with $a = 7, 13, 19, 21$, we restrict ourselves to the cases of $n = 7m$, $13m$, $19m$, and $21m$ with $m = 1, 2, \ldots$ (58).

B.1. *Equivariance of bifurcation equation*

Our objective here is to demonstrate that Lösch's hexagons with $a = 7, 13, 19, 21$ can be understood as bifurcated solutions from bifurcation points of multiplicity 12. As it turns out, not every bifurcation point of multiplicity 12 serves for this possibility, but only if it is associated with a 12-dimensional irreducible representation (k, ℓ) in (52) and (53) with some special values of k and ℓ.

To be specific, we investigate the following cases:

$$(n, k, \ell) = (7m, 2m, m),\ (13m, 3m, m),\ (19m, 3m, 2m),\ (21m, 4m, m), \quad \text{(B.1)}$$

where $m = 1, 2, \ldots$, corresponding to some of Lösch's ten smallest hexagons. We define

$$\hat{n} = \frac{n}{m}, \quad \hat{k} = \frac{k}{m}, \quad \hat{\ell} = \frac{\ell}{m}, \quad \text{(B.2)}$$

to obtain

$$(\hat{n}, \hat{k}, \hat{\ell}) = (7, 2, 1),\ (13, 3, 1),\ (19, 3, 2),\ (21, 4, 1). \quad \text{(B.3)}$$

Note that $\hat{n}$, $\hat{k}$, and $\hat{\ell}$ are pairwise relatively prime and satisfy

$$\hat{n} = \hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2, \quad \text{(B.4)}$$

which plays a key role in the subsequent derivation.

The bifurcation equation for the group-theoretic bifurcation point of multiplicity 12 is a 12-dimensional equation over $\mathbb{R}$. This equation can be expressed as a six-dimensional complex-valued equation in complex variables $z_1, \ldots, z_6$ as

$$F_i(z_1, \ldots, z_6, \tau) = 0, \quad i = 1, \ldots, 6, \quad \text{(B.5)}$$

where

$$(z_1, \ldots, z_6, \tau) = (0, \ldots, 0, 0)$$

is assumed to correspond to the bifurcation point. We often omit τ in the subsequent derivation.

Since the group $\mathrm{D}_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n)$ is generated by the four elements r, s, p_1, p_2, the equivariance of the bifurcation equation to the group $\mathrm{D}_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n)$ is equivalent to the equivariance to the action of these four elements. Therefore, the equivariance condition for (B.5) can be written as

$$r: \ \overline{F_3(z_1, z_2, z_3, z_4, z_5, z_6)} = F_1(\overline{z_3}, \overline{z_1}, \overline{z_2}, \overline{z_5}, \overline{z_6}, \overline{z_4}), \quad \text{(B.6)}$$

$$\overline{F_1(z_1, z_2, z_3, z_4, z_5, z_6)} = F_2(\overline{z_3}, \overline{z_1}, \overline{z_2}, \overline{z_5}, \overline{z_6}, \overline{z_4}), \quad \text{(B.7)}$$

$$\overline{F_2(z_1, z_2, z_3, z_4, z_5, z_6)} = F_3(\overline{z_3}, \overline{z_1}, \overline{z_2}, \overline{z_5}, \overline{z_6}, \overline{z_4}), \quad \text{(B.8)}$$

$$\overline{F_5(z_1, z_2, z_3, z_4, z_5, z_6)} = F_4(\overline{z_3}, \overline{z_1}, \overline{z_2}, \overline{z_5}, \overline{z_6}, \overline{z_4}), \quad \text{(B.9)}$$

$$\overline{F_6(z_1, z_2, z_3, z_4, z_5, z_6)} = F_5(\overline{z_3}, \overline{z_1}, \overline{z_2}, \overline{z_5}, \overline{z_6}, \overline{z_4}), \quad \text{(B.10)}$$

$$\overline{F_4(z_1, z_2, z_3, z_4, z_5, z_6)} = F_6(\overline{z_3}, \overline{z_1}, \overline{z_2}, \overline{z_5}, \overline{z_6}, \overline{z_4}); \quad \text{(B.11)}$$

$$s: \ F_{i+3}(z_1, z_2, z_3, z_4, z_5, z_6) = F_i(z_4, z_5, z_6, z_1, z_2, z_3),$$
$$F_i(z_1, z_2, z_3, z_4, z_5, z_6) = F_{i+3}(z_4, z_5, z_6, z_1, z_2, z_3), \quad i = 1, 2, 3; \quad \text{(B.12)}$$

$$p_j: \ \omega_{ji} F_i(z_1, \ldots, z_6) = F_i(\omega_{j1} z_1, \ldots, \omega_{j6} z_6), \quad j = 1, 2; \quad i = 1, \ldots, 6, \quad \text{(B.13)}$$

where

$$(\omega_{11}, \ldots, \omega_{16}) = (\omega^k, \omega^\ell, \omega^{-k-\ell}, \omega^k, \omega^\ell, \omega^{-k-\ell}),$$
$$(\omega_{21}, \ldots, \omega_{26}) = (\omega^\ell, \omega^{-k-\ell}, \omega^k, \omega^{-k-\ell}, \omega^k, \omega^\ell).$$

We expand F_1 as

$$F_1(z_1, z_2, z_3, z_4, z_5, z_6) = \sum_{a=0} \sum_{b=0} \cdots \sum_{u=0} A_{abcdeghijstu}(\tau) \times z_1^a z_2^b z_3^c z_4^d z_5^e z_6^g \overline{z_1}^h \overline{z_2}^i \overline{z_3}^j \overline{z_4}^s \overline{z_5}^t \overline{z_6}^u. \quad \text{(B.14)}$$

Since $(z_1, z_2, z_3, z_4, z_5, z_6, \tau) = (0, 0, 0, 0, 0, 0, 0)$ corresponds to the bifurcation point of multiplicity 12, we have

$$A_{000000000000}(0) = 0, \tag{B.15}$$

$$A_{100000000000}(0) = A_{010000000000}(0) = \cdots = A_{000000000001}(0). \tag{B.16}$$

The equivariance conditions (B.6)–(B.8) with respect to r give

$$\begin{aligned} F_1(z_1, z_2, z_3, z_4, z_5, z_6) &= \overline{F_2(\overline{z_3}, \overline{z_1}, \overline{z_2}, \overline{z_5}, \overline{z_6}, \overline{z_4})} \\ &= F_3(z_2, z_3, z_1, z_6, z_4, z_5) \\ &= \overline{F_1(\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4}, \overline{z_5}, \overline{z_6})}, \end{aligned}$$

from which we see that $A_{ab\cdots tu}$ are real. Then $F_2, \ldots, F_6$ are obtained from the equivariance conditions (B.6)–(B.11) and (B.12) with respect to r and s as

$$F_2(z_1, z_2, z_3, z_4, z_5, z_6) = F_1(z_2, z_3, z_1, z_6, z_4, z_5), \tag{B.17}$$

$$F_3(z_1, z_2, z_3, z_4, z_5, z_6) = F_1(z_3, z_1, z_2, z_5, z_6, z_4), \tag{B.18}$$

$$F_4(z_1, z_2, z_3, z_4, z_5, z_6) = F_1(z_4, z_5, z_6, z_1, z_2, z_3), \tag{B.19}$$

$$F_5(z_1, z_2, z_3, z_4, z_5, z_6) = F_1(z_5, z_6, z_4, z_3, z_1, z_2), \tag{B.20}$$

$$F_6(z_1, z_2, z_3, z_4, z_5, z_6) = F_1(z_6, z_4, z_5, z_2, z_3, z_1). \tag{B.21}$$

Next we determine the set of indices $(a, b, \ldots, t, u)$ of nonvanishing coefficients $A_{ab\cdots tu}(\tau)$ in (B.14). The equivariance conditions (B.13) with respect to p_1 and p_2 yield

$$\begin{aligned} &k(a-h) + \ell(b-i) - (k+\ell)(c-j) \\ &\quad + k(d-s) + \ell(e-t) - (k+\ell)(g-u) \\ &\quad \equiv k \mod n, \end{aligned} \tag{B.22}$$

$$\begin{aligned} &\ell(a-h) - (k+\ell)(b-i) + k(c-j) \\ &\quad - (k+\ell)(d-s) + k(e-t) + \ell(g-u) \\ &\quad \equiv \ell \mod n, \end{aligned} \tag{B.23}$$

which are equivalent, by (B.2), to

$$\begin{aligned} &\hat{k}(a-h) + \hat{\ell}(b-i) - (\hat{k}+\hat{\ell})(c-j) \\ &\quad + \hat{k}(d-s) + \hat{\ell}(e-t) - (\hat{k}+\hat{\ell})(g-u) \\ &\quad \equiv \hat{k} \mod \hat{n}, \end{aligned} \tag{B.24}$$

$$\begin{aligned} &\hat{\ell}(a-h) - (\hat{k}+\hat{\ell})(b-i) + \hat{k}(c-j) \\ &\quad - (\hat{k}+\hat{\ell})(d-s) + \hat{k}(e-t) + \hat{\ell}(g-u) \\ &\quad \equiv \hat{\ell} \mod \hat{n}. \end{aligned} \tag{B.25}$$

By introducing

$$(m_1, m_2, m_3) = (\hat{k},\ \hat{\ell},\ -(\hat{k}+\hat{\ell})) \tag{B.26}$$

we can rewrite (B.24) and (B.25) as

$$\begin{aligned} &(m_1, m_2, m_3, m_1, m_2, m_3) \\ &\quad \cdot (a-h, b-i, c-j, d-s, e-t, g-u) \\ &\quad \equiv m_1 \mod \hat{n}, \end{aligned} \tag{B.27}$$

$$\begin{aligned} &(m_2, m_3, m_1, m_3, m_1, m_2) \\ &\quad \cdot (a-h, b-i, c-j, d-s, e-t, g-u) \\ &\quad \equiv m_2 \mod \hat{n}, \end{aligned} \tag{B.28}$$

where "$\cdot$" denotes the inner product of vectors. We denote by S as the set of indices $(a, b, \ldots, t, u)$ that satisfy the above conditions, i.e.

$$S = \{(a, b, \ldots, t, u) \mid \text{(B.27) and (B.28)}\}. \tag{B.29}$$

Then $(a, b, \ldots, t, u)$ must belong to S if $A_{ab\cdots tu}(\tau) \neq 0$, and the converse is also true generically, except for the cases described in (B.15) and (B.16). Hence (B.14) can be replaced by

$$\begin{aligned} &F_1(z_1, z_2, z_3, z_4, z_5, z_6) \\ &\quad = \sum_S A_{abcdeghijstu}(\tau) \\ &\qquad \times z_1^a z_2^b z_3^c z_4^d z_5^e z_6^g \overline{z_1}^h \overline{z_2}^i \overline{z_3}^j \overline{z_4}^s \overline{z_5}^t \overline{z_6}^u. \end{aligned} \tag{B.30}$$

We observe here two facts that we need in Sec. B.2. The first fact is:

$$\begin{aligned} &a + b + c + h + i + j \\ &\quad \geq 1 \quad \text{for every } (a, b, \ldots, t, u) \in S. \end{aligned} \tag{B.31}$$

To see this we calculate $[(\text{B.24}) \times (\hat{k}+\hat{\ell}) + (\text{B.25}) \times \hat{k}]$ using (B.4), to obtain

$$\begin{aligned} &(\hat{k}^2 + 2\hat{k}\hat{\ell})(a-h) + (\hat{\ell}^2 - \hat{k}^2)(b-i) \\ &\quad - (2\hat{k}\hat{\ell} + \hat{\ell}^2)(c-j) \equiv \hat{k}^2 + 2\hat{k}\hat{\ell} \mod \hat{n}. \end{aligned}$$

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Hence we must have

$$(a, b, c, h, i, j) \neq (0, 0, 0, 0, 0, 0),$$

since $\hat{k}^2 + 2\hat{k}\hat{\ell} \not\equiv 0 \mod \hat{n}$ for the parameter values in (B.3). The second fact is:

$$\begin{aligned}&(m_3, m_1, m_2, m_2, m_3, m_1)\\ &\qquad \cdot (a-h, b-i, c-j, d-s, e-t, g-u)\\ &\quad \equiv m_3 \mod \hat{n},\end{aligned} \tag{B.32}$$

which results from the addition of (B.24) and (B.25).

B.2. *Bifurcated solutions*

For the bifurcation equation (B.5) we show the presence of bifurcated solutions such that

$$|z_1| = |z_2| = |z_3|, \quad z_4 = z_5 = z_6 = 0. \tag{B.33}$$

As their conjugate solutions, there also exist bifurcated solutions with

$$z_1 = z_2 = z_3 = 0, \quad |z_4| = |z_5| = |z_6|. \tag{B.34}$$

Although we do not exclude the possibility of other bifurcated solutions, those bifurcated solutions in (B.33) and (B.34) are sufficient for our purpose since they correspond to Lösch's hexagons with $a = 7, 13, 19, 21$, as we see below.

In the following, we focus on the solutions with $|z_1| = |z_2| = |z_3|$ and $z_4 = z_5 = z_6 = 0$ in (B.33). Such solutions satisfy $F_4 = F_5 = F_6 = 0$, since (B.19)–(B.21) together with (B.31) imply

$$F_4(z_1, z_2, z_3, 0, 0, 0) = F_1(0, 0, 0, z_1, z_2, z_3) = 0,$$

$$F_5(z_1, z_2, z_3, 0, 0, 0) = F_1(0, 0, 0, z_3, z_1, z_2) = 0,$$

$$F_6(z_1, z_2, z_3, 0, 0, 0) = F_1(0, 0, 0, z_2, z_3, z_1) = 0.$$

On the other hand, we see from (B.30) that

$$F_1(z_1, z_2, z_3, 0, 0, 0) = \sum_P A_{abc000hij000}(\tau) z_1^a z_2^b z_3^c \overline{z_1}^h \overline{z_2}^i \overline{z_3}^j, \tag{B.35}$$

where

$$P = \{(a, b, c, h, i, j) \mid (a, b, c, 0, 0, 0, h, i, j, 0, 0, 0) \in S\}.$$

For any $(a, b, c, h, i, j) \in P$ we have

$$(m_1, m_2, m_3) \cdot (a-h, b-i, c-j) \equiv m_1 \mod \hat{n}, \tag{B.36}$$

$$(m_2, m_3, m_1) \cdot (a-h, b-i, c-j) \equiv m_2 \mod \hat{n}, \tag{B.37}$$

$$(m_3, m_1, m_2) \cdot (a-h, b-i, c-j) \equiv m_3 \mod \hat{n} \tag{B.38}$$

by (B.27), (B.28) and (B.32). To find solutions for $F_1 = F_2 = F_3 = 0$, we set

$$z_j = \rho \exp(\mathrm{i}\theta_j) \qquad (j = 1, 2, 3).$$

Then, using (B.35) with (B.17) and (B.18), we obtain

$$\begin{aligned}
F_1(z_1, z_2, z_3, 0, 0, 0) &= \sum_P A_{abc000hij000}(\tau) z_1^a z_2^b z_3^c \overline{z_1}^h \overline{z_2}^i \overline{z_3}^j\\
&= \sum_P A_{abc000hij000}(\tau) \rho^{a+b+c+h+i+j} \exp \mathrm{i}[(\theta_1, \theta_2, \theta_3) \cdot (a-h, b-i, c-j)],\\
F_2(z_1, z_2, z_3, 0, 0, 0) &= F_1(z_2, z_3, z_1, 0, 0, 0)\\
&= \sum_P A_{abc000hij000}(\tau) z_2^a z_3^b z_1^c \overline{z_2}^h \overline{z_3}^i \overline{z_1}^j\\
&= \sum_P A_{abc000hij000}(\tau) \rho^{a+b+c+h+i+j} \exp \mathrm{i}[(\theta_2, \theta_3, \theta_1) \cdot (a-h, b-i, c-j)],\\
F_3(z_1, z_2, z_3, 0, 0, 0) &= F_1(z_3, z_1, z_2, 0, 0, 0)\\
&= \sum_P A_{abc000hij000}(\tau) z_3^a z_1^b z_2^c \overline{z_3}^h \overline{z_1}^i \overline{z_2}^j\\
&= \sum_P A_{abc000hij000}(\tau) \rho^{a+b+c+h+i+j} \exp \mathrm{i}[(\theta_3, \theta_1, \theta_2) \cdot (a-h, b-i, c-j)].
\end{aligned}$$

We consider two sets of solution candidates

$$(\theta_1,\theta_2,\theta_3)=\begin{cases}\dfrac{2\pi t}{\hat{n}}(m_1,m_2,m_3) & (t=0,1,\ldots,\hat{n}-1),\\[2mm] \dfrac{2\pi t}{\hat{n}}(m_1,m_2,m_3)+\pi(1,1,1) & (t=0,1,\ldots,\hat{n}-1).\end{cases} \tag{B.39}$$

For the first set $(\theta_1,\theta_2,\theta_3)=\frac{2\pi t}{\hat{n}}(m_1,m_2,m_3)$, we have

$$(\theta_1,\theta_2,\theta_3)\cdot(a-h,b-i,c-j)=\frac{2\pi t}{\hat{n}}(m_1,m_2,m_3)\cdot(a-h,b-i,c-j)\equiv\frac{2\pi t}{\hat{n}}m_1=\theta_1 \mod 2\pi$$

by (B.36). Therefore,

$$F_1=\rho\exp(\mathrm{i}\theta_1)\sum_{(a,b,c,h,i,j)\in P}A_{abc000hij000}(\tau)\rho^{a+b+c+h+i+j-1}.$$

Similarly, for F_2 and F_3, we use (B.37) and (B.38) to obtain

$$F_2=\rho\exp(\mathrm{i}\theta_2)\sum_{(a,b,c,h,i,j)\in P}A_{abc000hij000}(\tau)\rho^{a+b+c+h+i+j-1},$$

$$F_3=\rho\exp(\mathrm{i}\theta_3)\sum_{(a,b,c,h,i,j)\in P}A_{abc000hij000}(\tau)\rho^{a+b+c+h+i+j-1}.$$

Therefore,

$$\frac{F_1}{\rho\exp(\mathrm{i}\theta_1)}=\frac{F_2}{\rho\exp(\mathrm{i}\theta_2)}=\frac{F_3}{\rho\exp(\mathrm{i}\theta_3)}=\sum_{(a,b,c,h,i,j)\in P}A_{abc000hij000}(\tau)\rho^{a+b+c+h+i+j-1},$$

and the bifurcated solution curve is determined from

$$\sum_{(a,b,c,h,i,j)\in P}A_{abc000hij000}(\tau)\rho^{a+b+c+h+i+j-1}=0. \tag{B.40}$$

The leading terms of (B.40) are given as

$$A\tau+B\rho=0 \tag{B.41}$$

with generically nonzero coefficients A and B (see Remark 6.1). Equation (B.41) has a solution of the form $\rho=c\tau$ for some $c\neq 0$, which shows the generic existence of bifurcated solutions for all $(\theta_1,\theta_2,\theta_3)$ in (B.39).

For the second set $(\theta_1,\theta_2,\theta_3)=\frac{2\pi t}{\hat{n}}(m_1,m_2,m_3)+\pi(1,1,1)$ in (B.39), we have

$$\begin{aligned}(\theta_1,\theta_2,\theta_3)\cdot(a-h,b-i,c-j)&=\frac{2\pi t}{\hat{n}}(m_1,m_2,m_3)\cdot(a-h,b-i,c-j)+\pi(a+b+c-h-i-j)\\&\equiv\theta_1+\pi(a+b+c-h-i-j)\mod 2\pi\end{aligned}$$

by (B.36). Therefore,

$$F_1=\rho\exp(\mathrm{i}\theta_1)\sum_{(a,b,c,h,i,j)\in P}A_{abc000hij000}(\tau)(-1)^{a+b+c-h-i-j}\rho^{a+b+c+h+i+j-1}.$$

Likewise, we have

$$\frac{F_1}{\rho\exp(\mathrm{i}\theta_1)}=\frac{F_2}{\rho\exp(\mathrm{i}\theta_2)}=\frac{F_3}{\rho\exp(\mathrm{i}\theta_3)}=\sum_{(a,b,c,h,i,j)\in P}A_{abc000hij000}(\tau)(-1)^{a+b+c-h-i-j}\rho^{a+b+c+h+i+j-1},$$

and the bifurcated solution curve is determined from

$$\sum_{(a,b,c,h,i,j)\in P}A_{abc000hij000}(\tau)(-1)^{a+b+c-h-i-j}\rho^{a+b+c+h+i+j-1}=0. \tag{B.42}$$

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The leading terms of (B.42) are given as

$$-A\tau + B\rho = 0 \tag{B.43}$$

with generically nonzero coefficients A and B (see Remark 6.1).

Remark 6.1. The coefficients A and B in (B.41) and (B.43) are considered here. First note that $a + b + c + h + i + j \geq 1$ for all $(a, b, c, h, i, j) \in P$ with the equality holding only for $(a, b, c, h, i, j) = (1, 0, 0, 0, 0, 0)$. This shows $A = A'_{100000000000}(0)$, which denotes the derivative of $A_{100000000000}(\tau)$ with respect to τ, evaluated at $\tau = 0$. The other coefficient B is given as the sum of $A_{abc000hij000}(0)$ over all $(a, b, c, h, i, j) \in P$ with $a + b + c + h + i + j = 2$. We have $B = A_{000000011000}(0) + A_{001000100000}(0) + A_{020000000000}(0)$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (7, 2, 1)$ and $B = A_{000000011000}(0)$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (13, 3, 1), (19, 3, 2), (21, 4, 1)$.

B.3. *Symmetry of solutions*

To reveal the symmetry of the bifurcated solutions, we first consider the case of $(\theta_1, \theta_2, \theta_3) = (0, 0, 0)$ in (B.39). Then $z_1 = z_2 = z_3 = \rho \in \mathbb{R}$, whereas $z_4 = z_5 = z_6 = 0$. This solution, say, $z^{(0)} = (\rho, \rho, \rho, 0, 0, 0)$ is invariant to the action of r by (55), and hence the isotropy subgroup $\Sigma(z^{(0)})$ representing the symmetry of this solution contains $\langle r \rangle$. By (56), on the other hand, this solution has the symmetry of the form $p_1^{\alpha} p_2^{\beta}$ if and only if (α, β) satisfies the relations

$$k\alpha + \ell\beta \equiv 0,$$
$$\ell\alpha - (k + \ell)\beta \equiv 0,$$
$$-(k + \ell)\alpha + k\beta \equiv 0 \mod n.$$

By (B.2) and (B.4), this equation is satisfied by

$$(\alpha, \beta) = p(\hat{k} + \hat{\ell}, \hat{\ell}) + q(-\hat{\ell}, \hat{k}), \quad p, q \in \mathbb{Z}.$$

It therefore follows that $\Sigma(z^{(0)}) \supseteq \langle r, p_1^{\hat{k}+\hat{\ell}} p_2^{\hat{\ell}}, p_1^{-\hat{\ell}} p_2^{\hat{k}} \rangle$, where it can be verified that the inclusion is in fact equality, i.e.

$$\Sigma(z^{(0)}) = \langle r, p_1^{\hat{k}+\hat{\ell}} p_2^{\hat{\ell}}, p_1^{-\hat{\ell}} p_2^{\hat{k}} \rangle \tag{B.44}$$

$$= \begin{cases} \langle r, p_1^3 p_2, p_1^{-1} p_2^2 \rangle & ((\hat{n}, \hat{k}, \hat{\ell}) = (7, 2, 1)) \\ \langle r, p_1^4 p_2, p_1^{-1} p_2^3 \rangle & ((\hat{n}, \hat{k}, \hat{\ell}) = (13, 3, 1)) \\ \langle r, p_1^5 p_2^2, p_1^{-2} p_2^3 \rangle & ((\hat{n}, \hat{k}, \hat{\ell}) = (19, 3, 2)) \\ \langle r, p_1^5 p_2, p_1^{-1} p_2^4 \rangle & ((\hat{n}, \hat{k}, \hat{\ell}) = (21, 4, 1)). \end{cases} \tag{B.45}$$

The associated distributions correspond to Lösch's hexagons; indeed, for $(\alpha, \beta) = (\hat{k} + \hat{\ell}, \hat{\ell})$ or $(-\hat{\ell}, \hat{k})$, we have

$$\frac{T}{d} = \sqrt{\alpha^2 - \alpha\beta + \beta^2}$$
$$= \sqrt{\hat{k}^2 + \hat{k}\hat{\ell} + \hat{\ell}^2}$$
$$= \begin{cases} 7 & ((\hat{n}, \hat{k}, \hat{\ell}) = (7, 2, 1)) \\ 13 & ((\hat{n}, \hat{k}, \hat{\ell}) = (13, 3, 1)) \\ 19 & ((\hat{n}, \hat{k}, \hat{\ell}) = (19, 3, 2)) \\ 21 & ((\hat{n}, \hat{k}, \hat{\ell}) = (21, 4, 1)). \end{cases}$$

Let $z^{(t)}$ denote the solution corresponding to $(\theta_1, \theta_2, \theta_3) = \frac{2\pi t}{\hat{n}}(m_1, m_2, m_3)$ in (B.39), where $0 \leq t \leq \hat{n} - 1$. As shown in Table 3, we have

$$(m_1, m_2, m_3) \equiv \delta(\hat{\ell}, -\hat{k} - \hat{\ell}, \hat{k}) \mod \hat{n} \tag{B.46}$$

with $\delta = 2, 3, 11, 4$ for $\hat{n} = 7, 13, 19, 21$, respectively. Since $(\hat{\ell}, -\hat{k} - \hat{\ell}, \hat{k})$ corresponds to the action of p_2 on (z_1, z_2, z_3) in (56), $z^{(t)}$ is obtained from $z^{(0)}$ by the transformation of $p_2^{\delta t}$, which we may designate as $z^{(t)} = p_2^{\delta t} \cdot z^{(0)}$. Then the isotropy subgroup of $z^{(t)}$ is a conjugate subgroup of that of $z^{(0)}$, i.e.

$$\Sigma(z^{(t)}) = p_2^{\delta t} \cdot \Sigma(z^{(0)}) \cdot p_2^{-\delta t}.$$

This means that the solutions $z^{(t)}$ for $t \geq 1$ are essentially (or geometrically) the same as $z^{(0)}$.

A bifurcated solution of the form of (B.34), with $z_1 = z_2 = z_3 = 0$ and $|z_4| = |z_5| = |z_6|$, can be obtained from $z^{(0)}$ by transforming $z^{(0)}$ with s. The isotropy subgroup representing the symmetry of this solution is obtained as $s \cdot \Sigma(z^{(0)}) \cdot s^{-1}$. It is noted, however, that such conjugate solutions should be identified from a geometrical point of view.

Table 3. Value of δ in (B.46).

$\hat{n}$	$(\hat{k}, \hat{\ell})$	(m_1, m_2, m_3)	$\equiv$	$(\hat{\ell}, -\hat{k} - \hat{\ell}, \hat{k})$	$\times\ \delta$	
7	$(2, 1)$	$(2, 1, -3)$	$\equiv$	$(1, -3, 2)$	$\times\ 2$	mod 7
13	$(3, 1)$	$(3, 1, -4)$	$\equiv$	$(1, -4, 3)$	$\times\ 3$	mod 13
19	$(3, 2)$	$(3, 2, -5)$	$\equiv$	$(2, -5, 3)$	$\times\ 11$	mod 19
21	$(4, 1)$	$(4, 1, -5)$	$\equiv$	$(1, -5, 4)$	$\times\ 4$	mod 21