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Spatial period-doubling agglomeration of a core–periphery model with a system of cities

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ABSTRACT

The progress of spatial agglomeration of Krugman's core–periphery model is investigated by comparative static analysis of stable equilibria with respect to transport costs. We set forth theoretically possible agglomeration (bifurcation) patterns for a system of cities spread uniformly on a circle. A possible and most likely course predicted is a gradual and successive one, which is called spatial period doubling. For example, eight cities concentrate into four cities and then into two cities en route to the formation of a single city. The existence of this course is ensured by numerical simulation for the model. Such a gradual and successive agglomeration presents a sharp contrast to the agglomeration of two cities, for which spontaneous concentration to a single city is observed in core–periphery models of various kinds. Other bifurcations that do not take place in two cities, such as period tripling, are also observed. The need for study of a system of cities has thus been demonstrated.

1. Introduction

Emergence of spatial economic agglomeration attributable to market interactions has attracted much attention of spatial economists and geographers. Among many studies in the literature, the core–periphery model of Krugman (1991) is touted as the first and the most successful attempt to clarify the microeconomic underpinning of the spatial economic agglomeration in a full-fledged general equilibrium approach.[1] This model[2] introduced the Dixit and Stiglitz (1977) model of monopolistic competition into spatial economics and provided a new framework to explain interactions which occur among increasing returns at the level of firms, transportation costs, and factor mobility. Such a framework paved the way for the development of New Economic Geography[3] as a mainstream field of economics. Furthermore, in recent years, the framework has been applied to various policy issues in areas such as trade policy, taxation, and macroeconomic growth analysis (Baldwin et al., 2003).

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[1] This is based on an appraisal by Fujita and Thisse (2009) in honor of Krugman's 2008 Nobel Memorial Prize in Economic Sciences.

[2] This model considers an economy with two sectors (agriculture and manufacturing) and assumes an upper-tier utility function of the Cobb–Douglas type with CES sub-preferences over manufacturing varieties.

[3] See for overviews of these models: Brakman et al. (2001) and Combes et al. (2008).

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Yet most studies in New Economic Geography have remained confined to two-city models in which spatial economic concentration to a single city is triggered by bifurcation.[4] Although the two-city model is the most pertinent starting point by virtue of its analytical tractability, economic agglomerations, in reality, can take place at more than two locations, as corroborated by empirical evidence (Bosker et al., 2010). Among a system of cities, indirect spatial effects emerge and complicate the analysis. Behrens and Thisse (2007) stated that "Dealing with these spatial interdependencies constitutes one of the main theoretical and empirical challenges NEG and regional economics will surely have to face in the future." We must analyze a system of cities thoroughly in careful comparison with the two-city model to answer the question, "To what degree can we extrapolate the predictions and implications derived from two-city analysis to a system of cities?"

The agglomeration mechanism of a system of cities is yet to be untangled, although several attempts[5] have been conducted to transcend the two-city case.

- A local analysis (linearized eigenproblem) of the racetrack economy[6] has been conducted. Fujita et al. (1999) identified the emergence of several spatial frequencies. It is nonetheless difficult analytically to extract agglomeration properties from the nonlinear equations of core–periphery models with an arbitrary discrete number of cities.
- Numerical simulations have identified agglomeration patterns for systems of cities. A numerical simulation including 12 symmetric cities of equal size revealed that the symmetric equilibrium often becomes unstable (Krugman, 1993). Fujita et al. (1999) obtained post-bifurcation equilibria for three cities. Nevertheless, it seems premature to infer a global view of agglomeration based on currently available numerical information. A naïve numerical simulation for more cities might not be promising without a systematic methodology to investigate the resulting information.

In this paper, we investigate the orientation and progress of agglomerations for a multi-regional core–periphery model and, in turn, to test the adequacy of the two-city model as a spatial platform. Here we study comparative statics of (the set of) stable equilibria with respect to trade costs and not the time evolution of the state variable. Although Krugman (1993) and Fujita et al. (1999) give the orientation of the breaking of uniformity of the racetrack economy, we study the progress of agglomerations thereafter, along with the orientation. This paper provides a general framework for bifurcation analysis and presents the classification of a whole set of bifurcation patterns of the racetrack economy. We assume symmetry,[7] i.e., spatially uniform distribution of agricultural labors, homogeneous transportation costs, and so on. A model with symmetrically located places is an extreme case of equal competition among places and is of great interest to economists. Such a symmetric model undergoes bifurcation that breaks its symmetry, the mechanism of which is described by group-theoretic bifurcation theory.[8] The general framework given in this paper is, in principle, applicable to racetrack economy of core–periphery models of various kinds.

By group-theoretic bifurcation theory, possible agglomeration patterns occurring by bifurcations and courses of the pattern change of a racetrack economy, as the transport cost decreases, are obtained. A possible (and the most likely) predicted course of agglomeration is a *spatial period-doubling cascade* (see Proposition 5 in Section 4.3). An example of this cascade is presented for eight cities in Fig. 1. A system of $2^3=8$ identical cities (for some positive integer k) concentrates into $2^2=4$ identical larger cities, en route to the concentration to the single megalopolis. Consequently, the concentration progresses successively in association with the doubling of the spatial period.

The occurrence and non-occurrence of such bifurcations are dependent on individual cases for individual models and must be investigated for each case. We begin with the standard Krugman model as it is one of a few models readily formulated for a system of cities, and it is a topic in future to deal with other important core–periphery models, such as the Venables (1996) model and the general model by Puga (1999). The occurrence of bifurcations is investigated by computational bifurcation analysis. A combination of the group-theoretic bifurcation theory and the computational bifurcation theory is vital in the numerical simulation of the agglomerations of a racetrack economy with many cities. Yet this is not the only way for the investigation and the methodology proposed by Akamatsu et al. (2009) and Akamatsu and Takayama (2009)[9] serves as a possible alternative means for such investigation.

The possible equilibria and associated agglomeration patterns of 4, 6, 8, and 16 cities are studied theoretically (Section 4) and are obtained numerically (Section 5) in an exhaustive manner. Several new bifurcation behavioral characteristics of the racetrack economy, such as stable non-trivial equilibria, period-tripling, and a plethora of bifurcated paths, are obtained, in addition to the period-doubling cascade studied in Tabuchi and Thisse (2006,

[4] Two identical symmetric cities are in a stable state with high transport costs. When the costs are reduced to a certain level, a tomahawk bifurcation triggers a spontaneous concentration to a single city by breaking the symmetry (e.g., Krugman, 1991; Fujita et al., 1999; Forslid and Ottaviano, 2003).

[5] For recent contributions dealing with many locations, see, for example, Tabuchi et al. (2005) and Oyama (2009b).

[6] The racetrack economy uses a system of identical cities spread uniformly around the circumference of a circle. See, e.g., Krugman (1993, 1996), Fujita et al. (1999), Picard and Tabuchi (2010), and Tabuchi and Thisse (2006, 2011).

[7] Several recent papers (e.g., Berliant and Kung, 2009; Oyama, 2009a,b) describe the effects of asymmetries on the spatial agglomeration. Exploring multiple location model under such asymmetric assumption remains a task for future study.

[8] The major framework of this theory has already been developed in physics (see, e.g., Golubitsky et al., 1988; Ikeda and Murota, 2002), and is introduced in Section 3 in a manner applicable to the core–periphery model in Section 4.

[9] Akamatsu et al. (2009) proposed a methodology based on the discrete Fourier transformation of the spatial discounting matrix of the racetrack economy to demonstrate the occurrence of period-doubling bifurcation for $n=2^m$ cities. Akamatsu and Takayama (2009) applied this methodology to the four-city model for FO model (Forslid and Ottaviano, 2003).

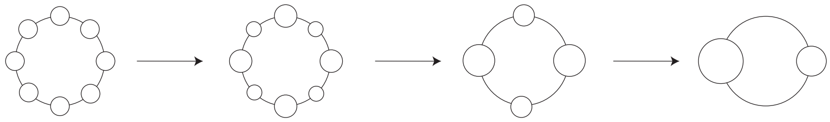

Fig. 1. Spatial period-doubling cascade for the eight cities (see Section 4.3 for the precise meaning of this figure; area of ○ denotes the size of the associated city; the arrow denotes the occurrence of a bifurcation).

2011).[10] The racetrack economy among a system of cities, for the same value of transport cost, has increasing quantities of stable equilibria when the number of cities increases. Such an increase of stable equilibria is a fundamental difficulty that might instill pessimism about the usefulness of the bifurcation analysis of the racetrack economy. Nonetheless, as the most likely course of agglomerations of a system of cities, the spatial period-doubling cascade for 4, 8, and 16 cities is actually found through numerical simulation, thereby resolving that pessimistic view.

This paper has several contributions over previous studies, such as Tabuchi and Thisse (2006, 2011), who predicted the period-doubling cascade: (1) this paper gives a general framework for bifurcation analysis and presents the classification of a whole set of bifurcation patterns of the racetrack economy, and (2) enumerates a whole set of bifurcation behaviors that involve several new characteristics, such as stable non-trivial equilibria, period-tripling, and a plethora of bifurcated paths.

This paper is organized as follows. The core–periphery model is introduced as a recapitulation and a reorganization of Krugman (1991) and Fujita et al. (1999, Chapter 5), and the governing equation is presented with a study of stability in Section 2. Group-theoretic bifurcation theory is presented in Section 3. The bifurcation theory of the racetrack economy is described in Section 4. Agglomerations of the racetrack economy with a system of cities are investigated numerically in Section 5. Appendices offer theoretical details and proofs.

2. Core–periphery model

A core–periphery model with an arbitrary discrete number of cities is presented as a recapitulation and a reorganization of Krugman (1991) and Fujita et al. (1999, Chapters 4 and 5) in a manner suitable for the theoretical analysis of the racetrack economy in Section 4 and the numerical analysis in Section 5.

2.1. Market equilibrium

The economy comprises n locations (labeled $i=1,\ldots,n$) around a circumference of a racetrack, two industrial sectors (agriculture and manufacturing), and two factors of production (agricultural labor and manufacturing labor). The agricultural sector is perfectly competitive and produces a homogeneous good, whereas the manufacturing sector is imperfectly competitive with increasing returns, producing horizontally differentiated goods. Laborers of each type consume two goods and supply one unit of labor inelastically. Agricultural laborers are immobile. Manufacturing laborers' distribution is expressed by $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)$, where λ_i is their population at the ith city. Based on an assumption of no population growth, $\boldsymbol{\lambda}$ is normalized by

$$\sum_{i=1}^{n} \lambda_i = 1, \quad \lambda_i \ge 0 \; (i=1,\ldots,n). \tag{1}$$

The domain of $\boldsymbol{\lambda}$, accordingly, is $(n-1)$-dimensional simplex. In the formulation of market equilibrium below, the spatial allocation of manufacturing workers is assumed to be given and $\boldsymbol{\lambda}$ is fixed and considered as a parameter, although they are allowed to migrate in the spatial equilibrium in Section 2.2.

Every consumer shares the same Cobb–Douglas tastes that are expressed by the utility function

$$U = M^{\mu} A^{1-\mu}, \tag{2}$$

where M means a composite index of the consumption of manufactured goods, A is the consumption of agricultural goods, and $\mu (0<\mu<1)$ is a constant representing the expenditure share of manufactured goods. The index M is defined using a constant-elasticity-of-substitution (CES) function:

$$M = \left[\int_0^k m(i)^{\rho}\, di \right]^{1/\rho} \quad (0<\rho<1), \tag{3}$$

where $m(i)$ denotes the consumption of each available variety, k is the range of varieties produced, and ρ is an index representing the intensity of the preference for variety in manufactured goods. We regard $\sigma = 1/(1-\rho) > 1$ as the elasticity

[10] Tabuchi and Thisse (2006, 2011) predicted that the racetrack economy of a core–periphery model could exhibit a spatial period-doubling bifurcation pattern. They also showed (by a numerical example) that a hierarchical urban system structure emerges for this model with multiple industries.

of substitution between differentiated goods. The consumer's problem is to maximize the utility (2) under the budget constraint.

The agricultural sector, which has constant returns to scale technology, requires one unit of unskilled labor in order to produce one unit of output. For simplicity, we assume that the agricultural-sector goods are transported freely between regions and are chosen as numéraire.

The manufacturing sector output is produced under increasing returns to scale and Dixit–Stiglitz monopolistic competition. Technologies, which are the same for all varieties and in all locations, involve a fixed input of $\mathcal{F}$ and marginal input requirement c^{M}.

We employ the *iceberg* transport technology, which implies that if a manufacturing variety produced at location i is sold at price p_i^{M}, then the delivered price p_{ij}^{M} of that variety at each consumption location j is given as $p_{ij}^{\mathrm{M}}=p_i^{\mathrm{M}}t_{ij}$, where $t_{ij}(>0)$ denotes the transport cost between the two cities i and j in terms of the amount of the manufactured good dispatched per unit received.

For the racetrack economy on a circle with the unit radius, which is studied in this paper, we define the transport cost t_{ij} as

$$t_{ij}=(1-\tau)^{-D_{ij}/\pi} \quad (i,j=1,\dots,n;\ 0<t_{ij};\ 0<\tau<1), \tag{4}$$

where τ is the transport parameter and

$$D_{ij}=\frac{2\pi}{n}\min(|i-j|,n-|i-j|)\geq 0 \quad (i,j=1,\dots,n) \tag{5}$$

represents the shortest distance between cities i and j along the arc; $\min(\cdot,\cdot)$ denotes the smaller value of the variables in parentheses. $\tau=0$ corresponds to the state of no transport cost, and $\tau=1$ corresponds to the state of infinite transport cost.

Some normalizations are introduced. For example, we choose units such that the marginal labor requirement satisfies

$$c^{\mathrm{M}}=\frac{\sigma-1}{\sigma}=\rho.$$

After some normalizations, the ratio of the manufacturing labor to the agricultural labor is set as $\mu:1-\mu$.

The market equilibrium conditions consist of the M-sector goods market clearing condition and the zero-profit condition due to the free entry and exit of firms. These conditions determine the income of each city, the price index of manufactures in that city, the wage rate of workers in that city, and the real wage rate in that city. After some normalizations the nominal wage rate w_i for the manufacturing labor force of the ith city is given as

$$w_i=\left[\sum_{j=1}^{n}Y_j t_{ij}^{1-\sigma}G_j^{\sigma-1}\right]^{1/\sigma} \quad (i=1,\dots,n); \tag{6}$$

and the manufacturers' price index for the ith city is given as

$$G_i=\left[\sum_{j=1}^{n}\lambda_j(w_j t_{ij})^{1-\sigma}\right]^{1/(1-\sigma)} \quad (i=1,\dots,n). \tag{7}$$

Here Y_i signifies the total income for the ith city, and λ_i $(0\leq\lambda_i\leq 1;\ i=1,\dots,n)$ stands for the ratio of the manufacturing labor force for the ith city to the whole manufacturing force, which is designated as the population of the ith city for short.

The total income for the ith city is expressed as

$$Y_i=\mu\lambda_i w_i+(1-\mu)/n \quad (i=1,\dots,n), \tag{8}$$

assuming that the agricultural wage has unity as numéraire, where the first term $\mu\lambda_i w_i$ on the right-hand-side of (8) is the income of the manufacturing labor force, and the second term $(1-\mu)/n$ is that of the agricultural force. The real wage ω_i of workers is defined as

$$\omega_i=w_i G_i^{-\mu} \quad (i=1,\dots,n). \tag{9}$$

Among many variables and parameters of these equations, we regard $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_n)^\top$ as an independent variable vector and τ as a bifurcation parameter.[11] Since the Krugman model is not analytically solvable, the real wages are to be obtained implicitly as $\omega_i=\omega_i(\boldsymbol{\lambda},\tau)$ $(i=1,\dots,n)$ from the set of (6)–(9). At the course of this, we assume a regularity condition on this set of such that $\omega_i=\omega_i(\boldsymbol{\lambda},\tau)$ $(i=1,\dots,n)$ turn out to be sufficiently smooth functions.

2.2. Spatial equilibrium

Following the market equilibrium, we introduce the spatial equilibrium, for which high skilled workers are allowed to migrate among cities. A customary way to define such an equilibrium is to consider the following

[11] μ and σ are regarded as auxiliary parameters that are pre-specified for each problem.

problem: find $(\boldsymbol{\lambda}^*,\hat{\omega})$ satisfying

$$\begin{cases} (\omega_i-\hat{\omega})\lambda_i^*=0, \quad \lambda_i^* \geq 0, \quad \omega_i-\hat{\omega} \leq 0 \quad (i=1,\ldots,n), \\ \sum_{i=1}^{n} \lambda_i^* = 1. \end{cases} \tag{10}$$

For the solution of this problem, $\hat{\omega}$ serves as the highest real wage. When the system is in a *spatial equilibrium* or a *sustainable equilibrium*, no individual can improve his/her real wage by changing his/her location unilaterally.

2.3. Stability

We march on to consider the stability of the spatial equilibrium satisfying (10), which can be considered as a population game. Among many alternatives to define the stability of the spatial equilibrium, we use the replicator dynamics,[12] which is most popular in the population game. This dynamics reads

$$\frac{d\boldsymbol{\lambda}}{dt} = \boldsymbol{F}(\boldsymbol{\lambda},\tau), \tag{11}$$

where

$$\overline{\omega} = \sum_{i=1}^{n} \lambda_i \omega_i \tag{12}$$

is the average real wage, $\boldsymbol{F}(\boldsymbol{\lambda},\tau) = (F_i(\boldsymbol{\lambda},\tau) | i=1,\ldots,n)$, and

$$F_i(\boldsymbol{\lambda},\tau) = (\omega_i(\boldsymbol{\lambda},\tau) - \overline{\omega}(\boldsymbol{\lambda},\tau))\lambda_i \quad (i=1,\ldots,n). \tag{13}$$

In this paper, we would like to replace a problem to obtain a set of stable spatial equilibria by another problem to find a set of stable stationary points of the replicator dynamics. Stationary points (rest points) $\boldsymbol{\lambda}^*(\tau)$ of the replicator dynamics (11) are defined as those satisfying the static governing equation

$$\boldsymbol{F}(\boldsymbol{\lambda}^*,\tau) = \boldsymbol{0}. \tag{14}$$

The relation $\sum_{i=1}^{n} \lambda_i^* = 1$ in (1) is always satisfied. These stationary points involve unsustainable points with $\lambda_i^* < 0$ and/or $\omega_i - \overline{\omega} > 0$ when they are unstable. Nonetheless stable stationary points are guaranteed to be sustainable by Proposition 1 below (Hofbauer and Sigmund, 1988).

Proposition 1. *A stable stationary point* $\boldsymbol{\lambda}^*(\tau)$ *of the dynamical system* (11) *with non-negative populations* $\lambda_i \geq 0$ $(i=1,\ldots,n)$ *is not only stable but also sustainable and, therefore, is a stable spatial equilibrium.*

Proof. See Appendix C.

To define stability of the stationary points,[13] we consider the Jacobian matrix

$$J(\boldsymbol{\lambda}^*,\tau) = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{\lambda}}(\boldsymbol{\lambda}^*,\tau) \tag{15}$$

of the governing equation.[14] If an equilibrium is linearly stable it is asymptotically stable, and if it is linearly unstable it is asymptotically unstable. We classify stability using the eigenvalues of this matrix:

$$\begin{cases} \text{linearly stable}: & \text{every eigenvalue has negative real part,} \\ \text{critical}: & \text{at least one eigenvalue is on the imaginary axis,} \\ \text{linearly unstable}: & \text{at least one eigenvalue has positive real part.} \end{cases}$$

3. Group-theoretic bifurcation theory

We have presented a set of static governing equations for the core–periphery model in Section 2. In this section, group-theoretic bifurcation theory is presented as a mathematical tool for describing bifurcation of the racetrack economy of this model. Symmetry of the racetrack economy is studied in Section 3.1. The symmetry of the governing equations is introduced to Section 3.2. The break bifurcation is investigated in Section 3.3.

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[12] Migration rules leading to the replicator dynamics are explained in Sandholm (2011) and its application in the context of economic geography is given in Oyama (2009b).

[13] For simplicity, the superscript $(\,)^*$ is suppressed in the following sections.

[14] The spatial equilibrium $\boldsymbol{\lambda}^*$ in (10) is defined on $(n-1)$-dimensional simplex. The extendibility of the Jacobian matrix for this simplex to the full n-dimension is presented in Sandholm (2011, Chapter 3).

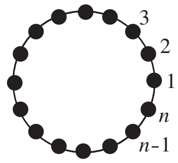


Fig. 2. Racetrack among n cities (n=16).

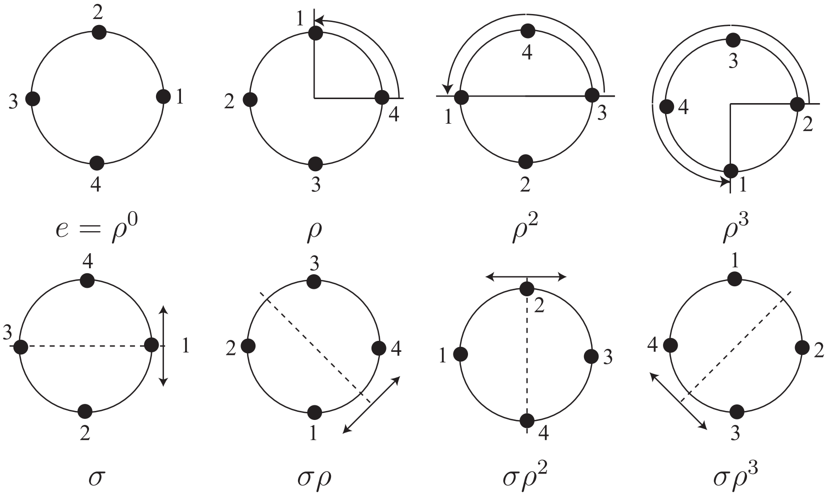


Fig. 3. Actions of elements of D_4 (e is the identity element).

3.1. Group expressing symmetry of racetrack economy

We consider the racetrack economy with n cities that are equally spread around the circumference of a circle as shown in Fig. 2, and describe the symmetry of these cities and of the governing equation by group.[15] Although the discussion in this section is applicable in principle to a general group G, we specifically examine a particular group: the dihedral group expressing the symmetry of a regular polygon.

Assumption 1 (*Parity*). We set n to be even. (The number of cities treated in the numerical analysis in Section 5 is n=4, 6, 8, and 16.)

The symmetry of these cities can be described by the dihedral group $G=D_n$ of degree n expressing regular n-gonal symmetry. This group is defined as

$$D_n=\{\rho^i,\sigma\rho^i \mid i=0,1,\ldots,n-1\},$$

where $\{\cdot\}$ denotes a group consisting of the geometrical transformations in the parentheses, ρ^i denotes a counterclockwise rotation about the center of the circle at an angle of $2\pi i/n$ ($i=0,1,\ldots,n-1$). In addition, $\sigma\rho^i$ is the combined action of the rotation ρ^i followed by the upside-down reflection σ (see Fig. 3 for n=4).

In our study of a system of n cities on the racetrack economy, each element g of D_n acts as a permutation among city numbers $(1,\ldots,n)$. Consequently, each representation matrix $T(g)$, which expresses the geometrical transformation by g, is a permutation matrix. With the use of the representation matrices for ρ and σ:

$$T(\rho)=\begin{pmatrix} & & & 1\\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix},\quad T(\sigma)=\begin{pmatrix} 1 & & & \\ & & & 1\\ & & \iddots & \\ & 1 & & \end{pmatrix},$$

the representation matrices $T(g)$ ($g\in D_n$) can be generated as

$$T(\rho^i)=\{T(\rho)\}^i,\quad T(\sigma\rho^i)=T(\sigma)\{T(\rho)\}^i\quad (i=0,1,\ldots,n-1).$$

[15] A group that consists of a set of geometrical transformations is used in the description of symmetry in various fields of science.

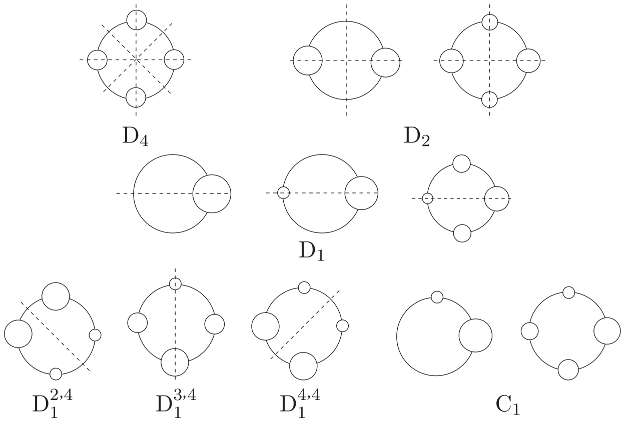


Fig. 4. Symmetries of equilibria for the four cities (n=4; dashed line, axis of reflection symmetry; the area of ○ signifies the population size).

Bifurcated equilibria from the D_n-symmetric racetrack economy have reduced symmetries that are labeled by subgroups[16] of D_n that denote partial symmetries of D_n. These subgroups are dihedral and cyclic groups that are given respectively as

$$D_m^{k,n} = \{\rho^{in/m}, \sigma\rho^{k-1+in/m} \,|\, i=0,1,\ldots,m-1\},$$

$$C_m = \{\rho^{in/m} \,|\, i=0,1,\ldots,m-1\}.$$

Therein, the subscript m $(=1,\ldots,n/2)$ is an integer that divides n. Superscript k $(=1,\ldots,n/m)$ expresses the directions of the reflection axes. Furthermore, C_m denotes cyclic symmetry at an angle of $2\pi/m$, and $D_m^{k,n}$ denotes reflection symmetry with respect to m-axes together with this cyclic symmetry.

In general, spatial distribution of populations with a higher symmetry with a larger m represents a more uniform state, while that with a lower symmetry with a smaller m represents a more concentrated state.

Example 1. The symmetries of equilibria, for example, for the four cities (n=4) are classified by these groups in Fig. 4. The patterns associated with the groups $D_1^{2,4}$ and $D_1^{4,4}$ have the same economic meaning. Such is also the case for the groups D_1 and $D_1^{3,4}$. □

Remark 1. In the interpretation of agglomeration patterns, we consider the spatial period along the unit circle of the racetrack economy. We define the spatial period for the D_n-invariant cities as $T_n = 2\pi/n$. When the cities are invariant under the transformation $\rho^{in/m}$, i.e., D_m- or $D_m^{k,n}$-invariant, we define the spatial period as $T_m = 2\pi/m$. In general, a higher spatial period with a larger m represents a more distributed spatial distribution of populations, a lower spatial period with a smaller m represents a more concentrated distribution.

3.2. Symmetry of governing equation

The nonlinear governing equation $\boldsymbol{F}(\boldsymbol{\lambda},\tau)$ in (14) of the racetrack economy is endowed with equivariance[17] with respect to $G = D_n$ (see Appendix C for proof):

$$T(g)\boldsymbol{F}(\boldsymbol{\lambda},\tau) = \boldsymbol{F}(T(g)\boldsymbol{\lambda},\tau), \quad g \in G = D_n \tag{16}$$

in terms of an $n \times n$ orthogonal matrix representation $T(g)$ of $G = D_n$ that expresses the geometrical transformation for an element g of $G = D_n$. The equivariance (16) means that if $(\boldsymbol{\lambda},\tau)$ is a solution to $\boldsymbol{F}(\boldsymbol{\lambda},\tau) = \boldsymbol{0}$, then so is $(T(g)\boldsymbol{\lambda},\tau)$.

In the description of the bifurcation of the racetrack economy, the equivariance of this economy is important as it paves the way for application of theory of break bifurcation in Section 3.3 for a particular case of $G = D_n$. The rule of hierarchical bifurcations in (22) for $G = D_n$ varies according to the value of the integer n (Appendix D.1).

Example 2. For the two-city model (n=2) with D_2-symmetry, we have

$$T(\rho) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{17}$$

[16] $D_{n/2}^{2,n}$-, C_n-, and $C_{n/2}$-symmetric modes are absent for this specific racetrack problem.

[17] The equivariance (16) is not an artificial condition for mathematical convenience, but is a natural consequence of the objectivity of the equation: the observer-independence of the mathematical description.

Then the equivariance condition (16) reduces to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}(\boldsymbol{\lambda},\tau) = \boldsymbol{F}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix},\tau\right).$$

The substitution of the explicit form $(F_i = (\omega_i(\boldsymbol{\lambda},\tau) - \overline{\omega}(\boldsymbol{\lambda},\tau))\lambda_i \mid i = 1,2)$ in (13) into this equation restricts expanded forms of $\omega_i(i = 1,2)$ as

$$\omega_1 = \sum_{a=0}\sum_{b=0} A_{ab}\lambda_1^a\lambda_2^b, \quad \omega_2 = \sum_{a=0}\sum_{b=0} A_{ab}\lambda_1^b\lambda_2^a$$

for some constants A_{ab}. At the equilibria $\lambda_1 = \lambda_2 = 1/2$ with the uniform population, we have symmetry conditions:

$$(\omega_1)^0 = (\omega_2)^0, \quad \left(\frac{\partial\omega_1}{\partial\lambda_1}\right)^0 = \left(\frac{\partial\omega_2}{\partial\lambda_2}\right)^0, \quad \left(\frac{\partial\omega_1}{\partial\lambda_2}\right)^0 = \left(\frac{\partial\omega_2}{\partial\lambda_1}\right)^0, \tag{18}$$

where $(\cdot)^0$ denotes that the associated term is evaluated at $\lambda_1 = \lambda_2 = 1/2$. These conditions are of great assistance in the investigation of possible bifurcation (see Example 5 in Section 4.2). □

As a consequence of (16), the Jacobian matrix in (15) is endowed with the symmetry condition

$$T(g)J(\boldsymbol{\lambda},\tau) = J(\boldsymbol{\lambda},\tau)T(g), \quad g \in G = \mathrm{D}_n \tag{19}$$

if $\boldsymbol{\lambda}$ is symmetric with respect to $G = \mathrm{D}_n$ in the sense that $T(g)\boldsymbol{\lambda} = \boldsymbol{\lambda}$ $(g \in G = \mathrm{D}_n)$. By virtue of (19), it is possible to construct a transformation matrix H such that the Jacobian matrix J is transformed into a block-diagonal form[18]:

$$\tilde{J} = H^\top J H = \begin{pmatrix} \tilde{J}_0 & & O \\ & \tilde{J}_1 & \\ O & & \ddots \end{pmatrix} \tag{20}$$

with diagonal block matrices $\tilde{J}_k$ $(k = 0,1,\ldots)$. Eigenvectors of the diagonal block $\tilde{J}_0$ are invariant to $G = \mathrm{D}_n$. Eigenvectors of other blocks $\tilde{J}_k$ $(k = 1,2,\ldots)$ have reduced symmetries labeled with subgroups G_k $(k = 1,2,\ldots)$ of $G = \mathrm{D}_n$. This is a mechanism to break symmetry via bifurcation. This block-diagonal form is suitable for an analytical eigenanalysis of the Jacobian matrix (Section 4.2).

Example 3. We consider the two-city model $(n=2)$ with D_2-symmetry. According to the symmetry conditions (19) with (17), the Jacobian matrix J^0, evaluated at the equilibria of uniform population $(\lambda_1 = \lambda_2 = 1/2)$, takes the form

$$J^0 = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{\lambda}}\bigg|_{\lambda_1 = \lambda_2 = 1/2} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

with ((13) and (18))

$$a = (\partial\omega_1/\partial\lambda_1 - 2\omega_1 - \partial\omega_2/\partial\lambda_1)^0/4, \quad b = (\partial\omega_2/\partial\lambda_1 - 2\omega_1 - \partial\omega_1/\partial\lambda_1)^0/4.$$

The transformation matrix H for block-diagonalization and the transformed Jacobian matrix $\tilde{J}^0$ are, respectively,

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \tilde{J}^0 = \begin{pmatrix} -(\omega_1)^0 & 0 \\ 0 & (\partial\omega_1/\partial\lambda_1 - \partial\omega_2/\partial\lambda_1)^0/2 \end{pmatrix}. \quad \square \tag{21}$$

3.3. Break bifurcation

The break bifurcation (Appendix B.4) is explained in light of group-theoretic bifurcation theory. This theory, a standard means to describe the bifurcation of symmetric systems, has been developed to obtain the rules of pattern formation—emergence of equilibria with reduced symmetries via so-called symmetry-breaking bifurcations (Golubitsky et al., 1988). This theory will be employed to investigate possible bifurcations of the racetrack economy in Section 4.

The bifurcation of a symmetric system with equivariance (16) has been studied in group-theoretic bifurcation theory for a general group G, including the dihedral group D_n, and has several properties described below (Ikeda and Murota, 2002).

- Property 1: The symmetry of the equilibrium points is preserved until branching into a bifurcated path.
- Property 2: The symmetry of equilibria on a bifurcated path is labeled by a subgroup, say G_1, of the group G.
- Property 3: In association with repeated bifurcations, one can find a hierarchy of subgroups

  $$G \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots \tag{22}$$

  that characterizes the hierarchical change of symmetries. Here $\longrightarrow$ denotes the occurrence of break bifurcation.
- Property 4: A bifurcated path sometimes regains symmetry on a bifurcation point on another equilibrium path with a higher symmetry.

[18] Theoretical details of the construction of the transformation matrix H are presented in Murota and Ikeda (1991) and Ikeda and Murota (2002).

Remark 2. In this section, the bifurcation rule is described in such a sequence that the symmetry is reduced successively via bifurcations. However, when we observe some economic system by decreasing the transport cost τ from 1 to 0, a bifurcated path sometimes regains symmetry at a bifurcation point, as explained in Property 4 presented above.

4. Bifurcation behavior for a racetrack economy

The tomahawk bifurcation of Krugman's core–periphery model with two cities is well known to produce spontaneous concentration to a single city. In contrast, it will be demonstrated in Section 5 that the racetrack economy of a system of cities displays more complex bifurcation.

The objective of this section is to present investigation of such bifurcation using the group-theoretic bifurcation theory presented in Section 3. We present several theoretical developments that will be used in the analysis of the racetrack economy in Section 5:

- Trivial equilibria[19] of the racetrack economy are determined in view of the symmetry in Section 4.1.
- Possible bifurcated equilibria and possible courses of bifurcations from the equilibrium of uniform population are investigated in Section 4.2.
- Among many possible equilibria predicted theoretically, a spatial period-doubling cascade is advanced as the most likely course en route to concentration in one city in Section 4.3.
- A systematic procedure to obtain equilibrium paths of the core–periphery model is presented in Section 4.4.

4.1. Trivial equilibria and sustain bifurcation

Trivial equilibria and sustain bifurcation on these equilibria are studied.

4.1.1. Trivial equilibria

The symmetry of the racetrack economy engenders trivial equilibria (Appendix B.2), which satisfy, for any values of τ, the nonlinear governing equation $\boldsymbol{F}(\boldsymbol{\lambda},\tau)=\boldsymbol{0}$ in (14). It is readily apparent that the racetrack economy has the *uniform-population trivial equilibrium*

$$\boldsymbol{\lambda}=(1/n,\ldots,1/n)^\top \tag{23}$$

with D_n-symmetry and with the spatial period of $T_n=2\pi/n$.

In addition to the equilibrium of uniform population in (23), several trivial equilibria exist as expounded in Proposition 2. The variety of trivial equilibria becomes diverse as the number n of cities increases.

Proposition 2 (*Period multiplying trivial equilibria*). *There are n/m trivial equilibria with*

$$\lambda_i=\begin{cases}1/m & (i=k,k+n/m,\ldots,k+(m-1)n/m),\\ 0 & \text{otherwise}\end{cases} \tag{24}$$

(m *divides* n; $k=1,\ldots,n/m$), *which, for example, for* $k=1$ *is* D_m*-symmetric. The spatial period* $T_m=2\pi/m$ *of these equilibria along the circle becomes* (n/m)*-times as long as the period* $T_n=2\pi/n$ *of the equilibrium of uniform population in* (23).

Proof. See Appendix C.

Among the trivial equilibria in Proposition 2, we are particularly interested in the following trivial equilibria:

- *Period-doubling trivial equilibria*

  $$\boldsymbol{\lambda}=(2/n,0,\ldots,2/n,0)^\top \quad \text{and } (0,2/n,\ldots,0,2/n)^\top \tag{25}$$

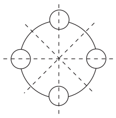
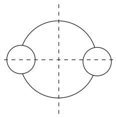
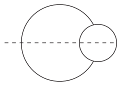
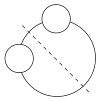
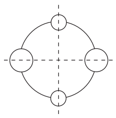
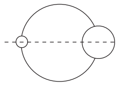
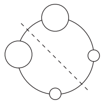
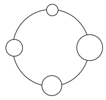
  express $D_{n/2}$-symmetric equilibria, for which a concentrating city and an extinguishing city alternate along the circle. The spatial period is $T_{n/2}=\pi/n$ and is doubled in comparison with the period $T_n=2\pi/n$ of the uniform population in (23). The two equilibria in (25) have the same economic meaning.
- *Concentrated trivial equilibria*

  $$\boldsymbol{\lambda}=(0,\ldots,0,\overset{i}{1},0,\ldots,0)^\top \quad (i=1,\ldots,n)$$

  express the concentration of the population to a single city. The equilibrium for $i=1$, for example, is D_1-symmetric.

[19] Trivial equilibria are those equilibria for which the population $\boldsymbol{\lambda}$ of the cities remains constant with respect to the change of the transport parameter τ (Appendix B.2).

Table 1
Examples of trivial and non-trivial equilibria for $n=4$ (dashed line, axis of reflection symmetry).

Group	D_4	D_2	D_1	$D_1^{2,4}$	C_1
Trivial					Non-existent
Non-trivial	Non-existent				

Example 4. Four cities ($n=4$) have trivial equilibria (see Table 1):

- the equilibrium of uniform population $\boldsymbol{\lambda}=(1/4,1/4,1/4,1/4)^\top$ (D_4-symmetry),
- the period-doubling trivial equilibrium $\boldsymbol{\lambda}=(1/2,0,1/2,0)^\top$ (D_2-symmetry),
- the concentrated trivial equilibrium $\boldsymbol{\lambda}=(1,0,0,0)^\top$ (D_1-symmetry),
- $D_1^{2,4}$-symmetric trivial equilibrium $\boldsymbol{\lambda}=(0,1/2,1/2,0)^\top$, and so on. □

4.1.2. Sustain bifurcation

For a trivial (corner) equilibrium with $\lambda_i=0$ for some i, the associated eigenvalue of the Jacobian matrix J is given as $e_i=\omega_i-\overline{\omega}$. The stability and criticality are classified accordingly[20] as (see (C.5)–(C.7) in Appendix C)

$$\begin{cases} \omega_i-\overline{\omega}>0 & \text{unstable (unsustainable)}, \\ \omega_i-\overline{\omega}=0 & \text{critical (sustain bifurcation)}. \end{cases}$$

For $\omega_i-\overline{\omega}<0$, whether the associated equilibria is stable or unstable depends on the signs of other eigenvalues of J.

4.2. Bifurcation from the uniform population

Bifurcation from the D_n-symmetric equilibrium of uniform population in (23) is investigated. Recall that n is assumed to be even.

According to the symmetry conditions (19), the Jacobian matrix J is a symmetric circulant matrix with entries

$$J_{ij}=k_l \quad (l=\min\{|i-j|,n-|i-j|\})$$

for some k_l ($l=1,2,\ldots$).

The transformation matrix H for block-diagonalization in (20) is given by (see Footnote 18)

$$H=\begin{cases} (\boldsymbol{\eta}^{(+)},\boldsymbol{\eta}^{(-)}) & \text{for } n=2, \\ (\boldsymbol{\eta}^{(+)},\boldsymbol{\eta}^{(-)},\boldsymbol{\eta}^{(1),1},\boldsymbol{\eta}^{(1),2},\ldots,\boldsymbol{\eta}^{(n/2-1),1},\boldsymbol{\eta}^{(n/2-1),2}) & \text{for } n\geq 4. \end{cases} \tag{26}$$

The column vectors of this matrix H, which will turn out to be the eigenvectors of J, are expressed as

$$\boldsymbol{\eta}^{(+)}=\frac{1}{\sqrt{n}}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}, \quad \boldsymbol{\eta}^{(-)}=\frac{1}{\sqrt{n}}\begin{pmatrix}\cos\pi\cdot 0\\ \vdots\\ \cos(\pi(n-1))\end{pmatrix}=\frac{1}{\sqrt{n}}\begin{pmatrix}1\\ -1\\ \vdots\\ 1\\ -1\end{pmatrix}, \tag{27}$$

$$\boldsymbol{\eta}^{(j),1}=\sqrt{\frac{2}{n}}\begin{pmatrix}\cos(2\pi j\cdot 0/n)\\ \vdots\\ \cos(2\pi j(n-1)/n)\end{pmatrix}, \quad \boldsymbol{\eta}^{(j),2}=\sqrt{\frac{2}{n}}\begin{pmatrix}\sin(2\pi j\cdot 0/n)\\ \vdots\\ \sin(2\pi j(n-1)/n)\end{pmatrix} \quad (j=1,\ldots,n/2-1). \tag{28}$$

[20] This classification is fundamentally identical to that explained in Fujita et al. (1999) for the two-city case.

These eigenvectors $\boldsymbol{\eta}^{(+)}$, $\boldsymbol{\eta}^{(-)}$, $\boldsymbol{\eta}^{(j),1}$, and $\boldsymbol{\eta}^{(j),2}$ are D_n-, $D_{n/2}$-, $D^{1,n}_{n/\widehat{n}}$-, and $D^{1+\widehat{n}/2,n}_{n/\widehat{n}}$-symmetric, respectively, and have respective spatial periods of $T_n = 2\pi/n$, $T_{n/2} = \pi/n$, $T_{n/\widehat{n}} = \widehat{n}\pi/n$, and $T_{n/\widehat{n}} = \widehat{n}\pi/n$. Here

$$\widehat{n} = n/\mathrm{gcd}(j,n) \geq 3, \tag{29}$$

and gcd(j,n) is the greatest common divisor of j and n.

The block-diagonal form in (20) reduces to a diagonal matrix as

$$\tilde{J} = H^\top J H = \mathrm{diag}(e^{(+)}, e^{(-)}, e^{(1)}, e^{(1)}, \ldots, e^{(n/2-1)}, e^{(n/2-1)}),$$

where diag $(\cdot)$ denotes a diagonal matrix with the diagonal entries therein. The diagonal entries, which correspond to the eigenvalues of J, are

$$e^{(+)} = k_0 + k_{n/2} + 2\sum_{l=1}^{n/2-1} k_l, \tag{30}$$

$$e^{(-)} = k_0 + (-1)^{n/2} k_{n/2} + 2\sum_{l=1}^{n/2-1} (-1)^l k_l, \tag{31}$$

$$e^{(j)} = k_0 + \cos(\pi j) k_{n/2} + 2\sum_{l=1}^{n/2-1} \cos(2\pi j l/n) k_l \quad (j = 1, \ldots, n/2-1). \tag{32}$$

It is noteworthy that $e^{(+)}$ and $e^{(-)}$ are simple eigenvalues, and that $e^{(j)}$ $(j = 1, \ldots, n/2-1)$ are double eigenvalues that are repeated twice.

Critical points on the equilibrium of uniform population are classified as

$$\begin{cases} e^{(+)} = 0: & \text{limit point of } \tau \ (M=1), \\ e^{(-)} = 0: & \text{simple bifurcation point } (M=1), \\ e^{(j)} = 0: & \text{double bifurcation point } (M=2), \end{cases}$$

where M is the multiplicity of a critical point that is defined as the number of zero eigenvalues of J. These simple and double bifurcation points are break bifurcation points.

The simple bifurcation point with $e^{(-)} = 0$ corresponds to the spatial period-doubling bifurcation, which engenders an alternating equilibrium (see Proposition 3).

Proposition 3 (*Simple bifurcation*). *At the simple bifurcation point, which is either a pitchfork or tomahawk (supercritical or subcritical), we encounter a symmetry-breaking bifurcation* $D_n \longrightarrow D_{n/2}$. *Its critical eigenvector is given uniquely as a* $D_{n/2}$*-symmetric vector* $\boldsymbol{\eta}^{(-)}$ *of* (27) *with components of alternating signs expressing the bifurcation mode of spatial period doubling from* $T = 2\pi/n$ *to* π/n.

Proof. See Chapter 8 of Ikeda and Murota (2002). □

The double bifurcation point $e^{(j)} = 0$ for some j corresponds to the spatial period $\widehat{n}$-times bifurcation (see (29) for the definition of $\widehat{n}$), which engenders a more rapid concentration than the period-doubling bifurcation of the simple bifurcation point (see Proposition 4). Double bifurcation points with $e^{(j)} = 0$ are absent for the two cities with $n=2$ (see (26)). It is noteworthy that the bifurcation of the two cities represents a special case, although four or more cities generally have double bifurcation points and have more diverse bifurcation properties.

Proposition 4 (*Double bifurcation*). *At the double bifurcation point with* $e^{(j)} = 0$ *for some* j, *we encounter a symmetry-breaking bifurcation* $D_n \longrightarrow D_{n/\widehat{n}}$, *at which the spatial period becomes* $\widehat{n}$*-times* ($\widehat{n} \geq 3$ *by* (29)).

Proof. See Appendix C.

Example 5. Recall Example 3 for the two-city model ($n=2$) with D_2-symmetry. The transformation matrix H for block-diagonalization (see Footnote 18) and the transformed Jacobian matrix $\tilde{J}^0$, evaluated at the equilibria of uniform population ($\lambda_1 = \lambda_2 = 1/2$), are given by (21) as

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \tilde{J}^0 = \begin{pmatrix} -(\omega_1)^0 & 0 \\ 0 & (\partial\omega_1/\partial\lambda_1 - \partial\omega_2/\partial\lambda_1)^0/2 \end{pmatrix}.$$

The first diagonal entry $-(\omega_1)^0$ of $\tilde{J}^0$ remains negative. The second entry $(\partial\omega_1/\partial\lambda_1 - \partial\omega_2/\partial\lambda_1)^0/2$ can possibly become zero at a break bifurcation point with the critical eigenvector $\boldsymbol{\eta}^{(-)} = (1/\sqrt{2}, -1/\sqrt{2})^\top$.

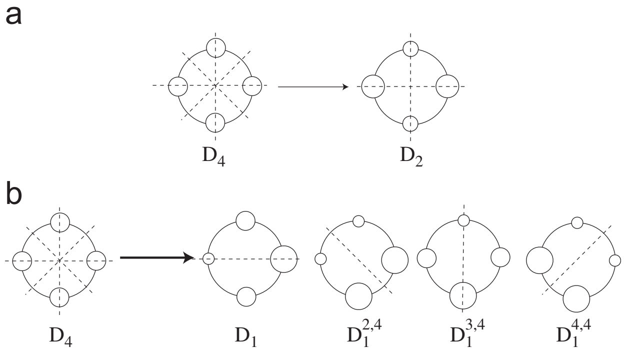


Fig. 5. Direct bifurcations from the four uniform cities (n=4; the arrow denotes the occurrence of a bifurcation). (a) Simple bifurcation. (b) Double bifurcation.

The stability and criticality for the equilibria of uniform population with $\lambda_1 = \lambda_2 = 1/2$ are classified[21] accordingly as

$$\begin{cases} (\partial\omega_1/\partial\lambda_1)^0 > (\partial\omega_2/\partial\lambda_1)^0 & \text{unstable (ordinary point)},\\ (\partial\omega_1/\partial\lambda_1)^0 = (\partial\omega_2/\partial\lambda_1)^0 & \text{critical (break bifurcation point)},\\ (\partial\omega_1/\partial\lambda_1)^0 < (\partial\omega_2/\partial\lambda_1)^0 & \text{stable (ordinary point)}. \end{cases}$$

If $(\partial\omega_1/\partial\lambda_1)^0 > (\partial\omega_2/\partial\lambda_1)^0$, when λ_1 increases, the rate of increase of the wage rate ω_1 for the city 1 is greater than that of ω_2 for city 2. Therefore, such an increase is accelerated and the equilibrium of uniform population with $\lambda_1 = \lambda_2 = 1/2$ becomes unstable. If $(\partial\omega_1/\partial\lambda_1)^0 = (\partial\omega_2/\partial\lambda_1)^0$, then an increase of λ_1 has identical influence on city 1 and city 2. This is the condition for break bifurcation. If $(\partial\omega_1/\partial\lambda_1)^0 < (\partial\omega_2/\partial\lambda_1)^0$, then an increase of λ_1 is more beneficial for city 2. Therefore, such an increase is damped and the equilibrium of uniform population with $\lambda_1 = \lambda_2 = 1/2$ remains stable. □

Example 6. The change of symmetry at bifurcation points is illustrated in Fig. 5 for the four cities (n=4). At the simple bifurcation point in Fig. 5(a), the bifurcation doubles the spatial period and triggers concentration of the population to two cities located at opposite sides of the circle, whereas the populations of the other two cities decline. At the double bifurcation point in Fig. 5(b) associated with $e^{(j)} = 0$ (n=4, j=1, $\hat{n} = 4$), the spatial period becomes four times. □

4.3. Spatial period-doubling cascade

As well as break bifurcations from the uniform-population trivial equilibrium with D_n-symmetry (Section 4.2), further break bifurcations may be encountered on (a) bifurcated paths of this D_n-symmetric equilibrium and (b) D_m-symmetric trivial equilibria (m divides n) presented in Section 4.1.

All these break bifurcations can be described using group-theoretic bifurcation theory (Ikeda and Murota, 2002). The rule of bifurcation depends on the integer number n: to be precise, the divisors of the number n. The bifurcation becomes increasingly hierarchical and complex for n with more divisors (Appendix D).

For D_n-symmetric cities with $n = 2^k$ (k is some positive integer), among a plethora of possible courses of hierarchical bifurcations, we devote special attention to the *spatial period-doubling bifurcation cascade* (Proposition 5)[22]:

$$D_{2^k} \longrightarrow D_{2^{k-1}} \longrightarrow D_{2^{k-2}} \longrightarrow \cdots \longrightarrow D_1, \qquad (33)$$

in which the spatial period is doubled successively by repeated simple bifurcations. Fig. 6 depicts this bifurcation for $n = 8 = 2^3$ cities.

Proposition 5 (*Spatial period-doubling cascade*). *The spatial period-doubling bifurcation cascade in* (33) *is potentially existent for* D_n*-symmetric cities with* $n = 2^k$ *for some integer* k. (*An actual existence of this cascade depends on individual cases with particular value of* n *and particular parameter values*.)

Proof. Proposition 3 gives $D_n \longrightarrow D_{n/2}$ for n even. Repeated use of this proposition for $n = 2^k, 2^{k-1}, \ldots, 2$ proves (33). □

Remark 3. Proposition 5 serves as a generalization of the study of Tabuchi and Thisse (2006, 2011) who conducted a local analysis (linearized eigenproblem) for the flat distribution of the racetrack economy to predict the occurrence of the period-doubling cascade.

[21] This classification is fundamentally identical to that presented by Fujita et al. (1999).

[22] A repeated doubling of the time period by bifurcations takes place in many physical systems (Feigenbaum, 1978) and is called period-doubling cascade.

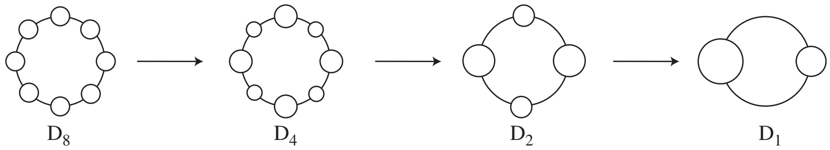


Fig. 6. Spatial period-doubling cascade for the eight cities ($n=8$; the arrow denotes the occurrence of a bifurcation).

4.4. Systematic procedure to obtain equilibrium paths

We present a systematic procedure to obtain equilibrium paths of the core–periphery model. First, we conduct an exhaustive search by obtaining all the equilibrium paths using the following steps:

Step 1: Obtain all trivial equilibria using the method presented in Section 4.1.
Step 2: Carry out the eigenanalysis of the Jacobian matrix J on these trivial equilibria to obtain the bifurcation points and to classify stable and unstable equilibria. For the equilibrium of uniform population, the formulas (30)–(32), which give the eigenvalues analytically, are to be used. The numerical eigenanalysis is to be conducted for other trivial equilibria.
Step 3: Obtain bifurcated paths branching from all these trivial equilibria using the computational bifurcation theory in Appendix E. The numerical eigenanalysis is to be conducted to find critical points and to investigate the stability of these equilibria.
Step 4: Repeat Steps 3 and 4 to exhaust all equilibrium paths.

Next, among all these equilibrium paths we select stable ones that are to be encountered when the transport cost τ is decreased from 1 to 0. The existence and multiplicity of possible stable ones for a particular value of τ must be investigated individually because they depend on the number n of cities, the values of the parameters σ and μ, and so on.

5. Numerical analysis of racetrack economy

Agglomerations of the racetrack economy of the core–periphery model are investigated for $n=4$, 6, 8, and 16 cities by comparative static analysis of equilibria with respect to transport costs using the systematic procedure to obtain equilibrium paths (Section 4.4). The agglomeration progresses via successive breaking of symmetries associated with successive elongation of the spatial period in agreement with the theoretical rule of bifurcation (Section 4). Successive and gradual progress of agglomerations by the spatial period-doubling cascade in Proposition 5 in Section 4.3 is highlighted as a key phenomenon for $n=4$, 8, and 16 cities in Section 5.1. The period-doubling and period-tripling are observed for $n=6$ cities in Section 5.2.

We set the elasticity of substitution as $\sigma=10.0$ and the ratio of the manufacturing labor force as $\mu=0.4$. These parameter values satisfy the so-called no-black-hole condition: $(\sigma-1)/\sigma=0.9>\mu=0.4$ (Fujita et al., 1999).

5.1. Period-doubling cascade

We demonstrate the occurrence of a period-doubling cascade for $n=4$, 8, and 16 cities (see Proposition 5 in Section 4.3).

5.1.1. Four cities

For the four cities ($n=4$), equilibrium paths were obtained using the systematic procedure to obtain equilibrium paths in Section 4.4. Fig. 7(a) shows τ versus λ_1 curves for these paths, which are apparently complex. Stable equilibria (shown as solid curves) and unstable ones (as dotted curves) are classified. Trivial equilibrium paths with D_m-symmetries ($m=1,2,4$) exist at the horizontal lines at $\lambda_1=0$, 1/4, 1/2, and 1, and several bifurcated paths connect these trivial paths.

To support the economic interpretation, among such complex paths, we have chosen stable trivial paths and associated paths shown in Fig. 7(b) as those most likely to occur; distributions of populations are portrayed at several equilibrium points. Stable parts (shown as solid lines) of the trivial equilibria are

- OA: equilibrium of uniform population $\boldsymbol{\lambda}=(1/4,1/4,1/4,1/4)^\top$ (D_4-symmetry),
- BC: period-doubling equilibrium $\boldsymbol{\lambda}=(1/2,0,1/2,0)^\top$ (D_2-symmetry),
- EF: concentrated equilibrium $\boldsymbol{\lambda}=(1,0,0,0)^\top$ (D_1-symmetry), and
- E′F′: another concentrated equilibrium $\boldsymbol{\lambda}=(0,0,1,0)^\top$ (D_1-symmetry).

Note that EF and E′F′ are symmetric counterparts with the same economic meaning.

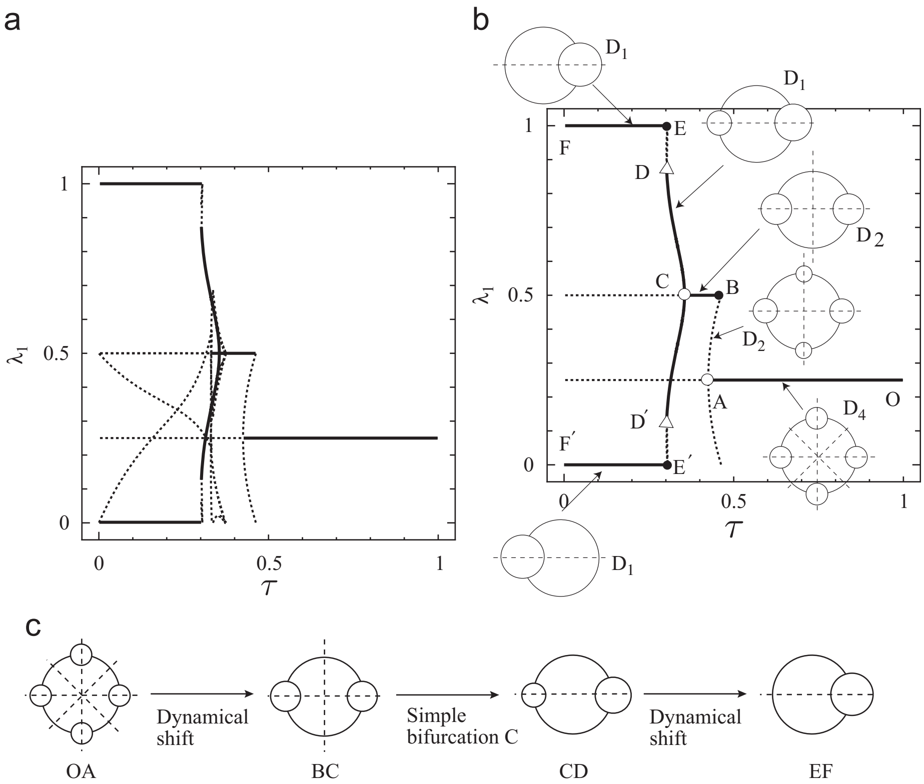


Fig. 7. Equilibrium paths of the four cities ($n=4$) and a predicted shift of stable equilibria in association with the decrease of transport cost (solid curve, stable; dashed curve, unstable; ○, simple break point; △, limit point; •, sustain point). (a) All equilibrium paths. (b) Stable trivial paths and associated paths. (c) Predicted shift of stable equilibria as τ decreases.

Bifurcation points on these trivial equilibria are classifiable as break and sustain points (Appendix B.4). Symmetries of the system are reduced at period-doubling breaking bifurcation points A and C denoted as ○ (Appendix D.1):

- At A, we encounter a symmetry breaking $D_4 \longrightarrow D_2$ associated with

$$\boldsymbol{\lambda} = (1/4, 1/4, 1/4, 1/4)^\top \rightarrow (1/4+\alpha, 1/4-\alpha, 1/4+\alpha, 1/4-\alpha)^\top \quad (|\alpha| < 1/4).$$

- At C, we encounter a symmetry breaking $D_2 \longrightarrow D_1$ associated with

$$\boldsymbol{\lambda} = (1/2, 0, 1/2, 0)^\top \rightarrow (1/2+\alpha, 0, 1/2-\alpha, 0)^\top \quad (|\alpha| < 1/2).$$

Symmetries are preserved at the sustain points denoted as •, at which a trivial equilibrium and a non-trivial one intersect (Appendix D.2). Sustain point B has D_2-symmetry; E and E′, D_1-symmetry. Consequently, the rule of break bifurcations in Fig. D1(a) in Appendix D.1 is of assistance in the tracing of bifurcated paths.

Among the bifurcated paths, we found the path CD and its symmetric counterpart CD′ to be stable. The stable path CD became unstable at the limit (maximum) point τ at D denoted by △ (see the left of Fig. B1(a) in Appendix B.3).

In view of the whole set of stable paths obtained herein, in association with the decrease of τ, we predict a possible course of the accumulation of population that follows four stable stages: OA, BC, CD, and EF, as presented in Fig. 7(c). Dynamical shifts[23] are assumed between OA and BC and between CD and EF. Starting from the uniform state $\boldsymbol{\lambda} = (1/4, 1/4, 1/4, 1/4)^\top$, via bifurcations and dynamical shifts, we arrive at the complete concentration $\boldsymbol{\lambda} = (1, 0, 0, 0)^\top$, in agreement with the rule of bifurcations in Fig. D1(a) in Appendix D.1. A spatial period-doubling cascade

$$D_4 \longrightarrow D_2 \longrightarrow D_1,$$

en route to the concentration to a single city occurs, in agreement with Proposition 5 in Section 4.3.

[23] When a stable equilibrium path becomes unstable at a critical point where a stable bifurcated path is non-existent, "dynamical shift" is assumed to take place to shift into another stable path.

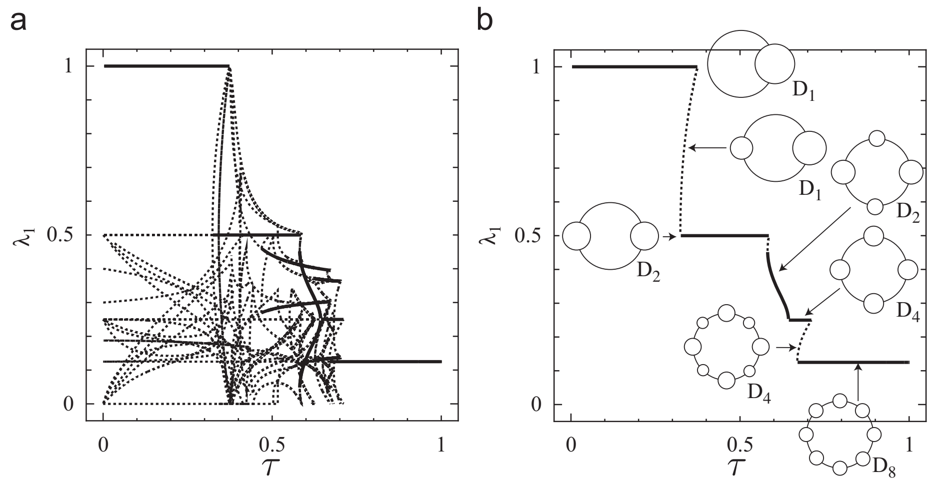


Fig. 8. Equilibrium paths of the eight cities expressed in terms of τ versus λ_1 curves ($n=8$; solid curve, stable; dashed curve, unstable). (a) All equilibrium paths. (b) Stable trivial paths and associated paths.

Recall that the stable equilibria of the simple tomahawk bifurcation[24] of the two cities consisted only of two trivial equilibria: the equilibrium of uniform population and the completely concentrated equilibrium. Different from the two cities, the four cities have simple pitchfork bifurcation point C and a stable non-trivial equilibrium[25] CD, for which migration from one city to another occurs stably without undergoing bifurcation. Moreover, the progress of agglomeration of the four cities is much more complex than that of the spontaneous concentration of the two cities triggered by the simple tomahawk bifurcation. An important point of caution is that the experience of the two cities is not to be regarded as universal. It motivates us to conduct bifurcation analysis for many cities in the remainder of this section.

Remark 4. For a simple bifurcation point, whether it is (subcritical) tomahawk or (supercritical) pitchfork is determined according to the sign of the stability coefficient. The determination of this coefficient, which in general involves many nonlinear terms (Thompson and Hunt, 1973), is beyond the scope of this paper.

5.1.2. *Eight cities*

For the eight cities bifurcated paths branching from several trivial equilibria are obtained in an exhaustive manner as shown in τ versus λ_1 relation of Fig. 8(a). The horizontal lines at $\lambda_1=0$, 1/8, 1/4, 1/2, and 1 are trivial equilibria with D_m-symmetries ($m=1,2,4,8$); these bifurcated paths that connect these trivial equilibria have grown more complex than those for the four cities in Fig. 7(a).

Among all the equilibrium paths for the eight cities shown in Fig. 8(a), stable equilibrium paths that are expected to be followed in association with the decrease of τ are depicted in Fig. 8(b). The spatial period-doubling cascade

$$D_8 \longrightarrow D_4 \longrightarrow D_2 \longrightarrow D_1 \tag{34}$$

engenders concentration into four cities and then into two cities, en route to concentration to a single city.

Complex bifurcated paths connecting these trivial equilibria were found in Fig. 8(a). Such complexity notwithstanding, all these paths have been traced successfully by the systematic procedure to obtain equilibrium paths in Section 4.4. That fact demonstrates the usefulness of this procedure. One might feel pessimistic when observing the complexity of the bifurcation of the racetrack economy that will grow rapidly with the increase of the number n of cities. Nonetheless, we can alleviate that pessimism by addressing only the stable equilibria, as in the spatial period-doubling cascade in (34).

5.1.3. *Sixteen cities*

Similar to the four and eight cities, the 16 cities displayed the spatial period-doubling cascade, as shown in Fig. 9,

$$D_{16} \longrightarrow D_8 \longrightarrow D_4 \longrightarrow D_2 \longrightarrow D_1.$$

[24] The tomahawk bifurcation was observed, e.g., in Krugman (1991) and Fujita et al. (1999) for the present model, and in Forslid and Ottaviano (2003) for an analytically solvable model.

[25] A stable non-trivial equilibrium branching from a supercritical pitchfork was observed also by Pflüger (2004) for a simple, analytically solvable model for the two cities.

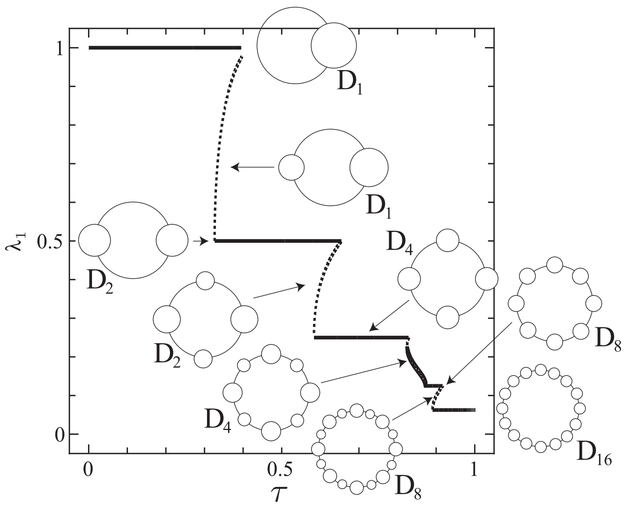


Fig. 9. Stable trivial paths and associated paths for the 16 cities that are expected to be followed in association with a decrease of τ ($n=16$; solid curve, stable; dashed curve, unstable).

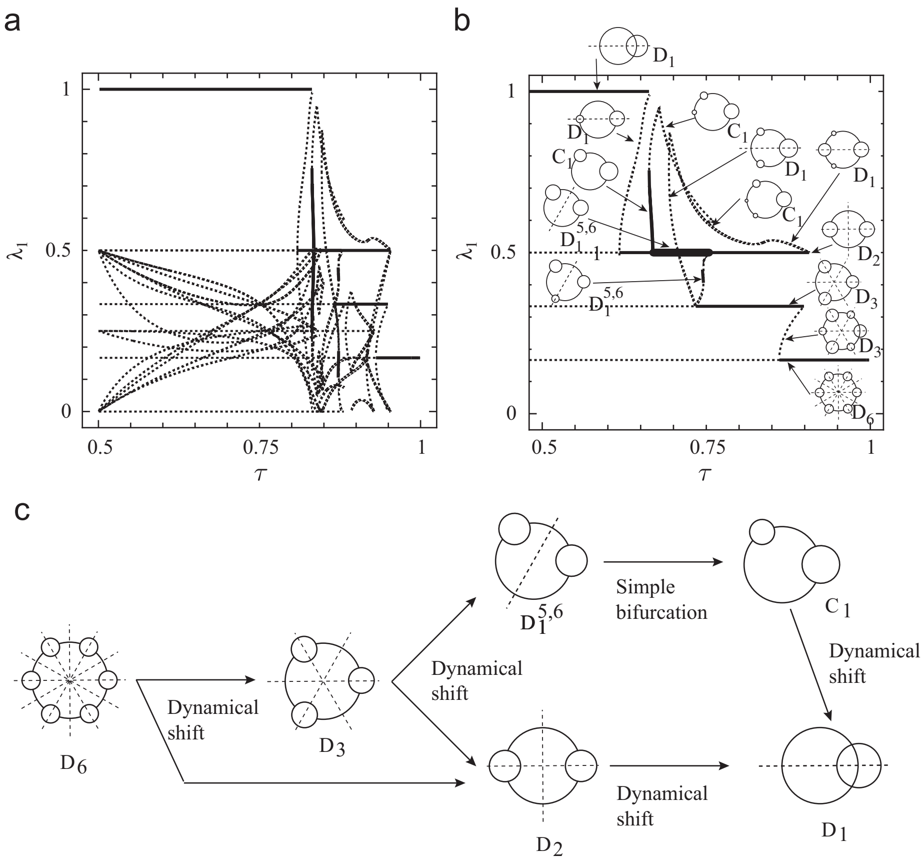


Fig. 10. Equilibrium paths of the six cities ($n=6$; solid curve, stable; dashed curve, unstable). (a) All equilibrium paths. (b) Stable trivial paths and associated paths. (c) Predicted shift of stable equilibria as τ decreases.

5.1.4. Discussion

The presence of the spatial period-doubling cascade, which is predicted by group-theoretic bifurcation theory in Proposition 5 and also in Tabuchi and Thisse (2006, 2011), has thus been ensured. This mechanism engenders concentration out of uniformity, especially for $n=2^k$ cities. It is to be remarked again that, unlike the two-city case, stable non-trivial equilibria exist.

For the 4, 8, and 16 cities, direct bifurcation is always a tomahawk (subcritical) and secondary bifurcation is always a pitchfork (supercritical). Such commonality is an interesting feature of the present model.

5.2. *Period doubling and tripling: six cities*

From the equilibrium paths of the six cities shown in Fig. 10(a), we chose stable paths and some associated paths shown in Fig. 10(b).

Trivial equilibria with D_m-symmetries ($m=1,2,3,6$) (Section 4.1) exist at the horizontal lines at $\lambda_1 = 0$, 1/6, 1/3, 1/4, 1/2, and 1:

- $\lambda_1 = 1/6$: D_6-symmetric equilibrium of uniform population,
- $\lambda_1 = 0$, 1/3: D_3-symmetric period-doubling trivial equilibria,
- $\lambda_1 = 0$, 1/4, 1/2: D_2-symmetric trivial equilibria, and
- $\lambda_1 = 0$, 1: D_1-symmetric concentrated trivial equilibria.

We observed

- period-doubling simple break bifurcations: $D_6 \longrightarrow D_3$ and $D_2 \longrightarrow D_1$;
- period-tripling double break bifurcations: $D_6 \longrightarrow D_2$ and $D_3 \longrightarrow D_1^{k,6}$.

This result arises from the fact that $n=6$ has two divisors: 2 and 3. Consequently, the period doubling is not as dominant as it is with the four cities (see Fig. 7(b)), but the period tripling via double break bifurcations plays an important role for $n=6$. The period tripling, which is theoretically predicted in Proposition 4 with $\hat{n} = 3$, does not take place for $n = 2^k$ cities, including the two cities.

A predicted shift of stable equilibria occurring in association with the decrease of τ is presented in Fig. 10(c). This shift is not unique:

- The D_6-symmetric state might dynamically shift into either the D_2- or D_3-symmetric state.
- The D_3-symmetric state might dynamically shift into the D_2- or $D_1^{5,6}$-symmetric state.

By virtue of the mixed occurrence of the period doubling and tripling, the predicted shift for the six cities with $n = 6 = 2 \times 3$ is more complex than that of the four cities portrayed in Fig. 7(b).

6. Conclusions

To verify the adequacy of the two-city case as a platform for spatial agglomeration, we investigated the progress of agglomeration of racetrack economy of the core–periphery model with 4, 6, 8, and 16 cities. These cities exhibited several features including the following:

- stable non-trivial equilibria,
- period-doubling cascade,
- a plethora of bifurcated paths, and
- period tripling via double break bifurcations.

These features were not observed and were overlooked in Krugman's two-city case, in which the tomahawk bifurcation engenders spontaneous concentration to a single city. It demands caution that the experience of the two-city case with a simple tomahawk bifurcation not be regarded as universal. It is preferable to employ a system of cities as a platform for the investigation of spatial agglomerations.

Symmetry-breaking bifurcation predicted by group-theoretic bifurcation theory proposed a broader view of the bifurcation of the racetrack economy. In fact, the bifurcation phenomena become progressively complex concomitantly with the increase of the number of cities. Such complexity might instill pessimism about the usefulness of the bifurcation analysis of the racetrack economy. Yet, when we specifically examine the economically stable equilibria that are expected to occur in association with the decrease of the transport cost, the spatial period-doubling cascade can be highlighted as the most likely mechanism to engender concentration out of uniformly distributed population. This suffices to resolve the pessimism.

Such complex phenomena can be traced in an exhaustive and systematic manner because of the insight of group-theoretic bifurcation theory. The proposed procedure is applicable to any new economic geography model other than Krugman's core–periphery model (see Footnote 3), and also to city distributions other than the racetrack economy. Accordingly, it will be an important topic of future studies to carry out bifurcation analysis of other updated new economic geography models with a system of cities scattered on a two-dimensional domain.

Acknowledgments

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Appendix A. Derivation of incremental equation

To derive the incremental equation for (14), we set

$$\boldsymbol{P}(\boldsymbol{\lambda},\boldsymbol{w},\tau)=\begin{pmatrix}(\omega_1-\overline{\omega})\lambda_1\\ \vdots \\ (\omega_n-\overline{\omega})\lambda_n\end{pmatrix}=\boldsymbol{0}, \tag{A.1}$$

here $\boldsymbol{w}=(w_1,\ldots,w_n)^\top$, $\omega_i=\omega_i(\boldsymbol{\lambda},\tau)$ $(i=1,\ldots,n)$, and $\overline{\omega}=\overline{\omega}(\boldsymbol{\lambda},\boldsymbol{w},\tau)=\sum_{i=1}^{n}\lambda_i\omega_i$ by (12). Eq. (6) is expressed as

$$\boldsymbol{M}(\boldsymbol{\lambda},\boldsymbol{w},\tau)=\begin{pmatrix}\displaystyle\sum_{s=1}^{n}Y_s(\boldsymbol{\lambda},\boldsymbol{w})t_{1s}^{1-\sigma}(\tau)G_s^{\sigma-1}(\boldsymbol{\lambda},\boldsymbol{w},\tau)-w_1^\sigma\\ \vdots \\ \displaystyle\sum_{s=1}^{n}Y_s(\boldsymbol{\lambda},\boldsymbol{w})t_{ns}^{1-\sigma}(\tau)G_s^{\sigma-1}(\boldsymbol{\lambda},\boldsymbol{w},\tau)-w_n^\sigma\end{pmatrix}=\boldsymbol{0}. \tag{A.2}$$

We rewrite (A.1) and (A.2) into incremental forms as

$$\frac{\partial\boldsymbol{P}}{\partial\boldsymbol{\lambda}}\delta\boldsymbol{\lambda}+\frac{\partial\boldsymbol{P}}{\partial\boldsymbol{w}}\delta\boldsymbol{w}+\frac{\partial\boldsymbol{P}}{\partial\tau}\delta\tau+\text{h.o.t.}=\boldsymbol{0}, \tag{A.3}$$

$$\frac{\partial\boldsymbol{M}}{\partial\boldsymbol{\lambda}}\delta\boldsymbol{\lambda}+\frac{\partial\boldsymbol{M}}{\partial\boldsymbol{w}}\delta\boldsymbol{w}+\frac{\partial\boldsymbol{M}}{\partial\tau}\delta\tau+\text{h.o.t.}=\boldsymbol{0} \tag{A.4}$$

(Ikeda and Murota, 2002, Chapter 7), where h.o.t. denotes higher-order terms. Under Assumption 2 below, we can eliminate independent variables $\delta\boldsymbol{w}$ from (A.3) and (A.4) to obtain

$$\delta\boldsymbol{w}=\left(\frac{\partial\boldsymbol{M}}{\partial\boldsymbol{w}}\right)^{-1}\left(\frac{\partial\boldsymbol{M}}{\partial\boldsymbol{\lambda}}J^{-1}\frac{\partial\boldsymbol{F}}{\partial\tau}-\frac{\partial\boldsymbol{M}}{\partial\tau}\right)\delta\tau,$$

and in turn to arrive at an incremental equilibrium equation

$$\tilde{\boldsymbol{F}}(\delta\boldsymbol{\lambda},\delta\tau)=J\delta\boldsymbol{\lambda}+\frac{\partial\boldsymbol{F}}{\partial\tau}\delta\tau+\text{h.o.t.}=\boldsymbol{0}, \tag{A.5}$$

where

$$J=\frac{\partial\boldsymbol{P}}{\partial\boldsymbol{\lambda}}-\frac{\partial\boldsymbol{P}}{\partial\boldsymbol{w}}\left(\frac{\partial\boldsymbol{M}}{\partial\boldsymbol{w}}\right)^{-1}\frac{\partial\boldsymbol{M}}{\partial\boldsymbol{\lambda}},\quad \frac{\partial\boldsymbol{F}}{\partial\tau}=\frac{\partial\boldsymbol{P}}{\partial\tau}-\frac{\partial\boldsymbol{P}}{\partial\boldsymbol{w}}\left(\frac{\partial\boldsymbol{M}}{\partial\boldsymbol{w}}\right)^{-1}\frac{\partial\boldsymbol{M}}{\partial\tau}.$$

Assumption 2 (*Regularity conditions*). The matrices J and $\partial\boldsymbol{M}/\partial\boldsymbol{w}$ are nonsingular.

Appendix B. Classifications and definitions of equilibria

In the study of the agglomeration of the core–periphery model, it is useful to resort to various kinds of classifications of equilibria.

B.1. Interior and corner equilibria

Equilibria of the present model are classifiable into two types:

- an interior equilibrium for which all cities have positive population $\lambda_i>0 (i=1,\ldots,n)$ and
- a corner equilibrium for which some cities have zero population.

The existence of the corner equilibrium is a special feature of the core–periphery model that demands reorganization in the application of bifurcation theory (Appendix D).

B.2. Trivial and non-trivial equilibria

The core–periphery model has characteristic equilibria, for which the population $\boldsymbol{\lambda}$ of the cities remains unchanged in association with the change of the transport parameter τ. We accordingly have the following classification:

$$\begin{cases} \text{Trivial equilibrium}: & \boldsymbol{\lambda} \text{ is constant with respect to } \tau. \\ \text{Non-trivial equilibrium}: & \boldsymbol{\lambda} \text{ is not constant with respect to } \tau. \end{cases}$$

B.3. Ordinary, limit, and bifurcation points

With reference to the eigenvalues e_i $(i=1,\ldots,n)$ of the Jacobian matrix J, equilibrium points are classified as

$$\begin{cases} e_i \neq 0 \text{ for all } i & \text{ordinary point,} \\ e_i = 0 \text{ for some } i & \text{critical (singular) point.} \end{cases}$$

Critical points are classified as

$$\begin{cases} \text{limit point of } \tau (M=1), \\ \text{bifurcation point} \begin{cases} \text{simple } (M=1), \\ \text{double } (M=2), \\ \vdots \end{cases} \end{cases}$$

Therein M is the multiplicity of a critical point that is defined as the number of zero eigenvalues of J.

At a limit point of τ, as portrayed in Fig. B1(a), the value of τ is maximized or minimized. A stable path is shown by the solid curve, and an unstable one by the dashed curve. Two half branches are connected at the limit point: a part of the path beyond the limit point is called a *half branch* and so is another part. Regarding stability, there are two cases:

- A half branch is stable, but another half branch is not.
- Both half branches are unstable.

At a bifurcation point, two or more equilibrium paths intersect: Fig. B1(b) presents a simple bifurcation point at which two paths (four half branches) intersect. Regarding stability, there are four cases: three, two, one, or zero half branches are stable; the remaining half branches are unstable. If we particularly examine only the stable half branches, they appear as a two-pronged weapon, a curve with a kink, a branch, and so on.

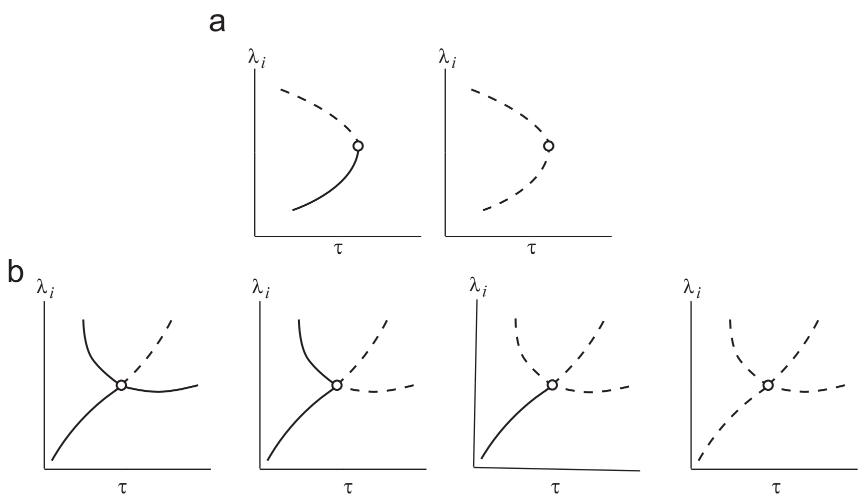

Fig. B1. Critical points (solid curve, stable; dashed curve, unstable). (a) Limit point of τ. (b) Bifurcation point.

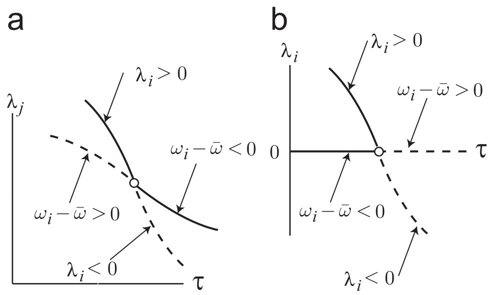

Fig. B2. Sustain points (solid curve, stable; dashed curve, unstable). (a) Crossing point of two non-trivial equilibria. (b) Crossing point of trivial and non-trivial equilibria.

B.4. Break and sustain points

Recall the block-diagonal form (20) in Section 3.3:

$$\tilde{J} = H^\top J H = \begin{pmatrix} \tilde{J}_0 & & O \\ & \tilde{J}_1 & \\ O & & \ddots \end{pmatrix}.$$

At a bifurcation point, a block $\tilde{J}_k$ for some k becomes singular. Dependent on the type of block that becomes singular, bifurcation points are classified into two types as described below.

- A *break bifurcation point*, or a break point, is symmetry-breaking one, at which $\tilde{J}_k$ becomes singular for some $k(\geq 1)$. The symmetry of the system is reduced on a bifurcated path (Section 4).
- A *sustain bifurcation point*, or a sustain point is a symmetry-preserving one, at which $\tilde{J}_0$ becomes singular. The symmetry of the system is preserved on a bifurcated path.

The sustain bifurcation point is an inherent feature of the present core–periphery model that permits the extinction of the population of manufacturing labor of a city. This point is necessarily a bifurcation point because the factorized form $(\omega_i - \overline{\omega})\lambda_i$ of (14) produces two independent equilibria. The point, as shown in Fig. B2, is classified into two types[26]: (a) the crossing point of two non-trivial equilibria and (b) the crossing point of a trivial equilibrium and a non-trivial equilibrium.

Since λ_i and $\omega_i - \overline{\omega}$ vanish simultaneously at this point, the sign of $\omega_i - \overline{\omega}$ along a (trivial) equilibrium path changes, as does the sign of λ_i along another path. At the point, a sustainable equilibrium $(\omega_i - \overline{\omega} < 0,\ \lambda_i = 0)$ changes into an unsustainable one $(\omega_i - \overline{\omega} > 0,\ \lambda_i = 0)$ along a path, whereas a stable equilibrium with positive population $(\lambda_i > 0,\ \omega_i - \overline{\omega} = 0)$ changes into an unstable one with negative population $(\lambda_i < 0,\ \omega_i - \overline{\omega} = 0)$.

Remark 5. Fujita et al. (1999) considered only sustainable equilibria, and regarded the sustain point as a kink that connects two half branches. Yet this point is regarded as a bifurcation point in this paper for consistency with the computational bifurcation theory in Appendix E.

Appendix C. Proofs

Proof of Proposition 1. The Jacobian matrix of the governing equation (14) reads

$$J = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{\lambda}} = \begin{pmatrix} \omega_1 - \overline{\omega} & 0 & \cdots & 0 \\ 0 & \omega_2 - \overline{\omega} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \omega_n - \overline{\omega} \end{pmatrix} + \begin{pmatrix} \Omega_{11}\lambda_1 & \Omega_{12}\lambda_1 & \cdots & \Omega_{1n}\lambda_1 \\ \Omega_{21}\lambda_2 & \Omega_{22}\lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \Omega_{n1}\lambda_n & \cdots & \cdots & \Omega_{nn}\lambda_n \end{pmatrix}$$
$$= \mathrm{diag}(\omega_1 - \overline{\omega}, \ldots, \omega_n - \overline{\omega}) + \mathrm{diag}(\lambda_1, \ldots, \lambda_n)\Omega, \tag{C.1}$$

where diag$(\cdots)$ denotes a diagonal matrix with diagonal entries therein and

$$\Omega_{ij} = \frac{\partial(\omega_i - \overline{\omega})}{\partial \lambda_j} \quad (i,j = 1, \ldots, n), \tag{C.2}$$

[26] The sustain point for the two cities in Fujita et al. (1999) corresponds to the crossing point of a trivial equilibrium and a non-trivial equilibrium in Fig. B2(b).

$$\Omega=(\Omega_{ij} \mid i,j=1,\ldots,n). \tag{C.3}$$

For an interior equilibrium with $\omega_i-\overline{\varpi}=0$ and $\lambda_i>0$ $(i=1,\ldots,n)$ (Appendix B.1), the Jacobian matrix in (C.1) reduces to

$$J=\text{diag}(\lambda_1,\ldots,\lambda_n)\Omega. \tag{C.4}$$

An interior equilibrium is stable if all the eigenvalues of the matrix Ω in (C.3) have negative real parts because the matrix diag $(\lambda_1,\ldots,\lambda_n)$ in (C.4) is positive definite.

A corner equilibrium (Appendix B.1) can be expressed without loss of generality[27] as

$$\lambda_i>0, \quad \omega_i-\overline{\varpi}=0 \quad (i=1,\ldots,m), \tag{C.5}$$

$$\lambda_i=0 \quad (i=m+1,\ldots,n). \tag{C.6}$$

With the use of (C.5), the Jacobian matrix in (C.1) becomes

$$J=\left(\begin{array}{c|ccc} \Phi_1 & & \Phi_2 & \\ \hline & \omega_{m+1}-\bar{\omega} & & 0 \\ O & & \ddots & \\ & 0 & & \omega_n-\bar{\omega} \end{array}\right), \tag{C.7}$$

where $\Phi_i=\text{diag}(\lambda_1,\ldots,\lambda_m)\Omega_i$ $(i=1,2)$ and

$$\Omega_1=\begin{pmatrix} \Omega_{11} & \cdots & \Omega_{1m} \\ \vdots & \ddots & \vdots \\ \Omega_{m1} & \cdots & \Omega_{mm} \end{pmatrix}, \quad \Omega_2=\begin{pmatrix} \Omega_{1(1+m)} & \cdots & \Omega_{1n} \\ \vdots & \ddots & \vdots \\ \Omega_{m(1+m)} & \cdots & \Omega_{mn} \end{pmatrix}.$$

By (C.7), $e_i=\omega_i-\overline{\varpi}$ $(i=m+1,\ldots,n)$ are eigenvalues of J, whereas the other m eigenvalues e_i $(i=1,\ldots,m)$ are given as eigenvalues of Φ_1.

For a stable corner equilibrium, we have $e_i=\omega_i-\overline{\varpi}<0$ $(i=m+1,\ldots,n)$, whereas $\omega_i-\overline{\varpi}=0$ $(i=1,\ldots,m)$ by (C.6). Therefore, the sustainability conditions $\omega_i-\overline{\varpi}\leq 0$ and $\lambda_i \geq 0$ $(i=1,\ldots,n)$ in (10) are satisfied for the stable equilibrium. Consequently, the check of the sustainability is to be replaced with the investigation of stability. □

Proof of the equivariance (16) for D_n. In our study of a system of n cities on the racetrack economy, each element g of D_n acts as a permutation among city numbers $(1,\ldots,n)$ (Section 3.1). By expressing the action of $g\in D_n$ as $g: i\mapsto i^*$ for city numbers i and i^*, for any $g\in D_n$, we have $\overline{\varpi}(T(g)\boldsymbol{\lambda},\tau)=\overline{\varpi}(\boldsymbol{\lambda},\tau)$ by (12); we also have $\omega_i(T(g)\boldsymbol{\lambda},\tau)=\omega_{i^*}(\boldsymbol{\lambda},\tau)$ because of the homogeneity of the transport cost ((4) with (5)). Therefore, we have

$$F_i(T(g)\boldsymbol{\lambda},\tau)=(\omega_{i^*}(\boldsymbol{\lambda},\tau)-\overline{\varpi}(\boldsymbol{\lambda},\tau))\lambda_{i^*}=F_{i^*}(\boldsymbol{\lambda},\tau)$$

by (13). This proves the equivariance (16). □

Proof of Proposition 2. We consider D_m-symmetric state, for which the equivariance (16) with $G=D_m$ for the explicit form of $\boldsymbol{F}$ in (14) entails

$$\begin{aligned} &\omega_1=\omega_{1+n/m}=\cdots=\omega_{1+(m-1)n/m}, \\ &\omega_2=\omega_{2+n/m}=\cdots=\omega_{2+(m-1)n/m}, \\ &\vdots \\ &\omega_{n/m}=\omega_{2n/m}=\cdots=\omega_n. \end{aligned} \tag{C.8}$$

As a candidate for a trivial equilibrium, we consider a D_m-symmetric population distribution

$$\begin{cases} \lambda_i=1/m,\ \omega_i-\overline{\varpi}=0 & (i=1,1+n/m,\ldots,1+(m-1)n/m), \\ \lambda_i=0 & \text{otherwise}, \end{cases} \tag{C.9}$$

which is obtained by setting $k=1$ in (24).

The substitution of (C.9) into (14) yields

$$\boldsymbol{F}=\begin{pmatrix} \{\omega_1-\overline{\varpi}\}\lambda_1 \\ \vdots \\ \{\omega_n-\overline{\varpi}\}\lambda_n \end{pmatrix}=\begin{pmatrix} 0\times\lambda_1 \\ (\omega_2-\omega_1)\times 0 \\ \vdots \\ (\omega_{n/m}-\omega_1)\times 0 \\ \vdots \end{pmatrix}=\boldsymbol{0}.$$

This proves that (C.9) is a trivial equilibrium, while other trivial equilibria are treated similarly. □

[27] Because all corner equilibria can be reduced to the form by appropriately rearranging the order of independent variables $\boldsymbol{\lambda}$, the consideration of this form does not lose generality.

Proof of Proposition 4. The critical eigenvector is given by the superposition of the two vectors $\boldsymbol{\eta}^{(j),1}$ and $\boldsymbol{\eta}^{(j),2}$ in (28) as

$$\boldsymbol{\eta}(\theta) = \cos\theta \cdot \boldsymbol{\eta}^{(j),1} + \sin\theta \cdot \boldsymbol{\eta}^{(j),2}$$

for general angle θ $(0 \le \theta < 2\pi)$. The bifurcated paths do not branch in the general direction associated with arbitrary θ, but branch in finite directions as expounded in Lemma 1 below. This novel aspect presented in this paper has not been reported to date for the core–periphery model. □

Lemma 1. *As made clear by group-theoretic analysis* (Ikeda et al., 1991; Ikeda and Murota, 2002), *bifurcated paths branching at the double bifurcation point satisfy the following properties*:

(i) *There exist $\widehat{n}$ bifurcated paths ($2\widehat{n}$ half branches) in the directions of*

$$\delta\boldsymbol{\lambda} = C\boldsymbol{\eta}(\alpha_k), \quad C\boldsymbol{\eta}(\alpha_{k+\widehat{n}}) \quad (k = 1, \ldots, \widehat{n}),$$

where C is a scaling constant and

$$\alpha_i = -\pi(i-1)/\widehat{n} \quad (i = 1, \ldots, 2\widehat{n}).$$

(ii) *The equilibria $\delta\boldsymbol{\lambda} = C\boldsymbol{\eta}(\alpha_k)$ and $C\boldsymbol{\eta}(\alpha_{k+\widehat{n}})$ are $\mathrm{D}^{k,n}_{n/\widehat{n}}$-symmetric $(k = 1, \ldots, \widehat{n})$. Therefore, the spatial period becomes $\widehat{n}$-times $(\widehat{n} \ge 3)$ in comparison with that of the D_n-symmetric equilibrium of uniform population.*

(iii) *The $2\widehat{n}$ half branches are classifiable into two independent ones: $\delta\boldsymbol{\lambda} = C\boldsymbol{\eta}(\alpha_{2l-1})$, $C\boldsymbol{\eta}(\alpha_{2l})$ $(l = 1, \ldots, \widehat{n})$. It suffices in numerical analysis to find the two branches in two directions: $\delta\boldsymbol{\lambda} = C\boldsymbol{\eta}(\alpha_1)$, $C\boldsymbol{\eta}(\alpha_2)$.* □

Remark 6. Proposition 1 is extendible to double bifurcation points on bifurcated paths with D_m-symmetry (m divides n; $m \ge 3$) by choosing $\boldsymbol{\eta}^{(j),1}$ and $\boldsymbol{\eta}^{(j),2}$ to be $\mathrm{D}^{1,m}_{m/\widehat{m}}$- and $\mathrm{D}^{1+\widehat{m}/2,m}_{m/\widehat{m}}$-symmetric, respectively. Here $\widehat{m} = m/\gcd(j,m)$ and $1 \le j < m/2$.

Appendix D. Hierarchical bifurcations

We explain the mechanism of hierarchical bifurcations of the racetrack economy that consist of symmetry-breaking at break points and the extinction of city population of manufacturing labor at sustain points, en route to the concentration of population in a city. As mentioned in B.4, these points are characterized by

$$\begin{cases} \text{break point}: & \text{symmetry breaking,} \\ \text{sustain point}: & \text{symmetry preserving.} \end{cases}$$

D.1. Break bifurcations

In addition to break bifurcations from the uniform-population trivial equilibrium with D_n-symmetry (Section 4.2), several possible sources of symmetry-breaking exist. Namely, further break bifurcations might be encountered on (a) bifurcated paths of this D_n-symmetric equilibrium and (b) D_m-symmetric trivial equilibria (m divides n) presented in Section 4.1.

All these break bifurcations can be described using group-theoretic bifurcation theory (Ikeda and Murota, 2002). The rule of bifurcation depends on the integer number n. To be precise, it depends on the divisors of the number n. The bifurcation becomes increasingly hierarchical and complex for n with more divisors.

A few examples are explained below.

- If n is a prime number, then it can undergo only one course of hierarchical bifurcations: $\mathrm{D}_n \longrightarrow \mathrm{D}_1 \longrightarrow \mathrm{C}_1$.
- For $n = 4 = 2^2$, a hierarchy of subgroups expressing the rule of hierarchical break bifurcations is presented in Fig. D1(a). As might be readily apparent, in addition to the direct bifurcations in Fig. 5, several secondary and tertiary bifurcations exist: D_2-symmetric equilibrium branches into $\mathrm{D}^{k,4}_1$-symmetric ones $(k = 1, \ldots, 4)$ and $\mathrm{D}^{k,4}_1$-symmetric one branches into C_1-symmetric one.
- The hierarchy of subgroups for $n = 6 = 2 \times 3$ shown in Fig. D1(b) portrays a more complex hierarchy than that of $n = 4 = 2^2$ in Fig. D1(a).

These rules are sufficient in the description for break bifurcations of the model, although the model in general undergoes more complex bifurcation attributable to the presence of sustain bifurcation points (Appendix D.2).

D.2. Sustain bifurcations

The sustain bifurcation point is a special feature of the core–periphery model that engenders the extinction of city population of manufacturing labor (Appendix B.4). If we follow only stable equilibria, we must switch to another stable

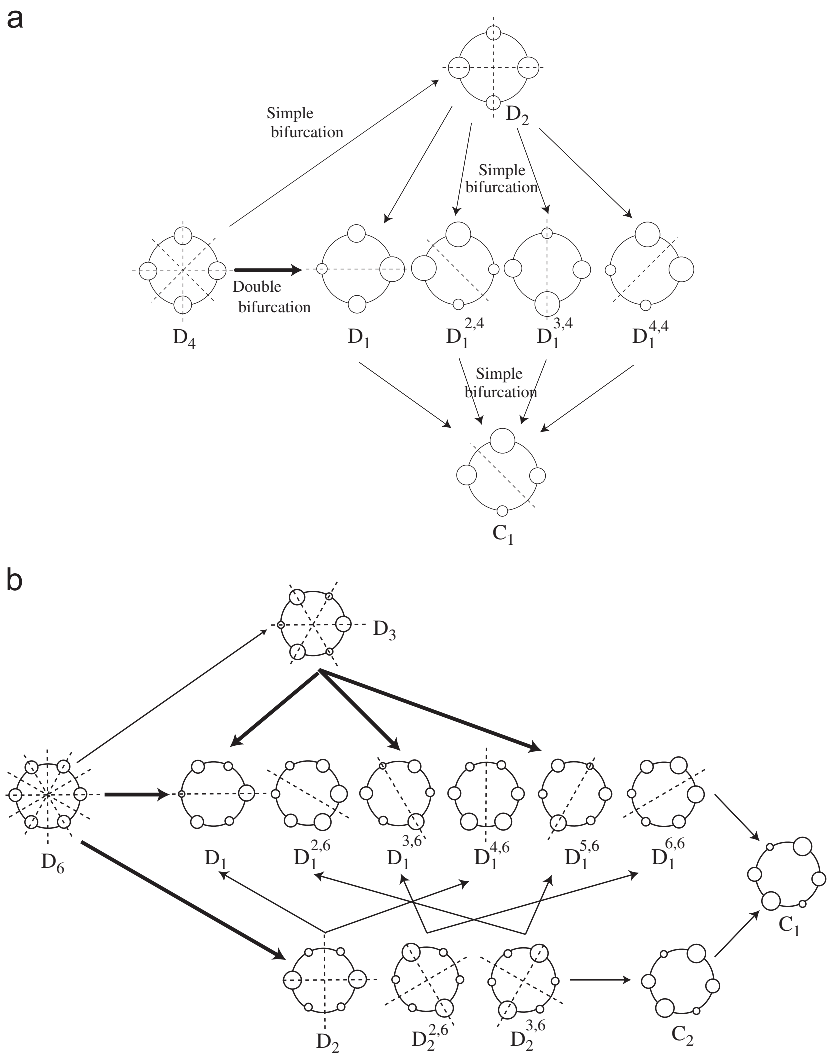

Fig. D1. Hierarchy of subgroups associated with hierarchical break bifurcations (thick solid arrow, double bifurcation; thin solid arrow, simple bifurcation). (a) Four cities ($n=4$). (b) Six cities ($n=6$).

path. A sustain point might appear in any equilibrium other than the equilibrium of uniform population. The presence of sustain bifurcation points must be meshed into the rule of break bifurcations (Appendix D.1).

A possible course of hierarchical bifurcations is presented in Fig. D2, for example, for the four cities ($n=4$). From the trivial equilibrium with uniform population ($\boldsymbol{\lambda}=(1/4,1/4,1/4,1/4)^{\top}$) shown at the left, population distribution patterns of various kinds are engendered via hierarchical bifurcations. This figure shows sustain bifurcation points indicated by the dashed arrows, in addition to the double bifurcation indicated by the thick solid arrow and the simple bifurcations indicated by the thin solid arrows. Cases other than $n=4$ can be treated similarly.

Appendix E. Computational bifurcation theory

As presented in Section 5, the equilibria of the governing equation of the racetrack economy involve several bifurcated paths that are quite complex. These paths can be traced in a systematic and exhaustive manner using computational bifurcation theory (Crisfield, 1977). Concretely speaking, we employ the following three numerical steps:

- *Path tracing*: In the path tracing of non-trivial equilibria, we refer to the incremental form (A.5) of the governing equation (14), i.e.,

$$\tilde{\boldsymbol{F}}(\delta\boldsymbol{\lambda},\delta\tau)=J(\boldsymbol{\lambda},\tau)\delta\boldsymbol{\lambda}+\frac{\partial\boldsymbol{F}}{\partial\tau}(\boldsymbol{\lambda},\tau)\delta\tau+\text{h.o.t.}=\boldsymbol{0}. \tag{E.1}$$

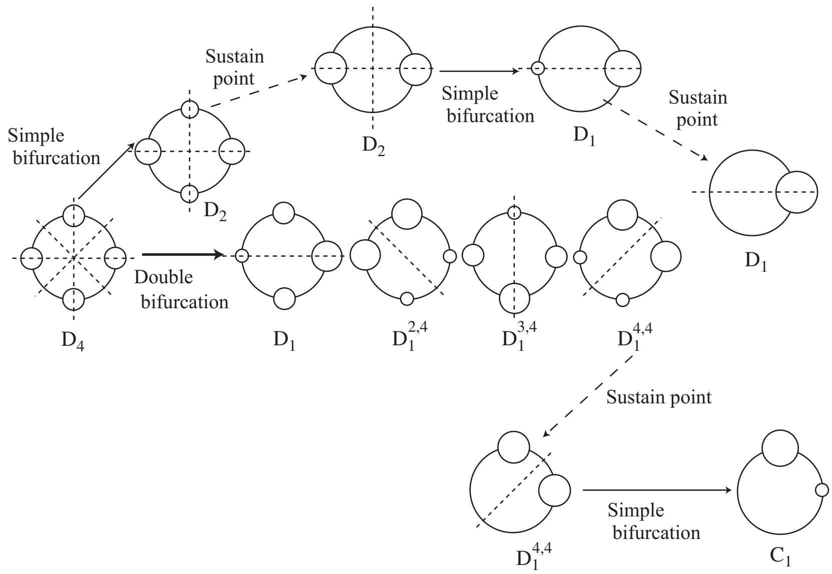

Fig. D2. Hierarchical, break, and sustain bifurcations of the four cities ($n=4$; thick solid arrow, double bifurcation; thin solid arrow, simple bifurcation; dashed arrow, sustain bifurcation).

At each equilibrium point $(\boldsymbol{\lambda},\tau)$, another equilibrium point $(\boldsymbol{\lambda}+\delta\boldsymbol{\lambda},\tau+\delta\tau)$ can be found by solving[28] (E.1) for $(\delta\boldsymbol{\lambda},\delta\tau)$ using a predictor–corrector (Newton–Raphson) type method.
For the trivial equilibria, we need not carry out path tracing because they are obtainable simply by symmetry consideration (Section 4.1), although singularity detection and branch switching must be conducted.

- *Singularity detection*: We carry out eigenanalysis of the Jacobian matrix $J(\boldsymbol{\lambda},\tau)$ at the equilibria $(\boldsymbol{\lambda},\tau)$ to find the location of a critical point, as a point at which one or more eigenvalues e_i of $J(\boldsymbol{\lambda},\tau)$ become zero.
For the equilibrium of uniform populations, the eigenanalysis can be conducted using the explicit formulas presented in (27)–(32). For the other equilibria, the numerical eigenanalysis of J, which is in general a non-symmetric matrix, must be conducted.
- *Branch switching*: To obtain bifurcated paths branching from a bifurcation point, we use the so-called line search method.
At a simple bifurcation point with a single zero eigenvalue, say $e_1=0$ $(M=1)$, a bifurcated path is to be sought in the direction of the critical eigenvector $\boldsymbol{\eta}_1$ associated with this zero eigenvalue. We employ $\delta\boldsymbol{\lambda}=C\boldsymbol{\eta}_1$ as the initial value for the iteration to find an equilibrium on a bifurcated path, where the scaling constant C is to be specified pertinently in view of the convergence of the iteration.
At a double bifurcation point with two zero eigenvalues, say $e_1=e_2=0$ $(M=2)$, two (independent) bifurcated paths are to be sought in the directions $\delta\boldsymbol{\lambda}=C\boldsymbol{\eta}(\alpha_1)$ and $C\boldsymbol{\eta}(\alpha_2)$ in Proposition 1(iii).

We carry out an *exhaustive search* of a whole set of equilibrium paths of the racetrack economy by repeating the three steps described above for secondary, tertiary, ..., bifurcated paths, until all possible bifurcated paths are exhausted.

For the racetrack economy, it is possible to trace trivial equilibria independently, other than the equilibrium of uniform population, and to find bifurcated paths from these trivial equilibria (Section 4.4).

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[28] In the incremental equation (E.1), $(\boldsymbol{\lambda},\tau)$ is fixed, $\delta\boldsymbol{\lambda}$ is treated as an independent variable, and $\delta\tau$ is treated as a bifurcation parameter.

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