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Dynamic Network Equilibrium Model of Simultaneous Route / Departure Time Choice for a Many-to-One OD Pattern

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Abstract

Dynamic equilibrium traffic assignment with elastic demand is analyzed on an over saturated network for a many-to-one origin-destination pattern. The model aims to obtain time-dependent cumulative arrival curves or equivalently arrival flow rates at each of the nodes explicitly taking into account the effects of queues. We assume that the OD demand is elastic; that is, the departure times of users should be determined through the assignment given time constraints at the single destination such as work starting times. We first show that the problem can be decomposed with respect to the arrival time at the destination. Alternative formulations for the problem are then shown as the Variational Inequality, Nonlinear Complementarity and Fixed Point problems using two kinds of unknowns: link flows and arrival times at nodes. Unlike the path-based formulation in the previous studies, the mapping used in the formulation is simple and, therefore, easy to analyze the mathematical properties of the assignment. Thus, the existence of the solution is established and the uniqueness conditions are examined based upon the formulations. Finally, solution algorithms using merit functions that enforce the global convergence are suggested.

1. Introduction

This research deals with the dynamic equilibrium traffic assignment on an over-saturated network with a many-to-one OD pattern taking simultaneously route and departure time choices into account, given time constraints at the single destination such as work starting times. The research purpose is to examine properties of the problem based on several alternative formulations for obtaining cumulative arrival curves at every node over a discrete network so as to simultaneously establish the equilibrium of route as well as departure time choices.

On the departure time choice for the morning/evening commute trips, Vickrey(1969), Hendrickson et al.(1981), Hurdle(1981), DePalma et al.(1983), Smith(1984) and Daganzo(1985) studied the single bottleneck problem; that is, they assumed that commuters with common travel cost function pass a single bottleneck, and their departure times are determined given their work schedules so as to establish an equilibrium. Kuwahara and Newell(1987) then extended the analysis so as to incorporate route choice as well with a many-to-one OD pattern, provided that everyone is assumed to pass through a bottleneck only once. Although they considered queuing delay (waiting time in a queue) explicitly, these studies have been restricted so far to a network with a limited number of bottlenecks. In particular, every commuter was assumed to pass a bottleneck only once in most studies in this category.

On the other hands, Kuwahara and Akamatsu (1993), Akamatsu and Kuwahara (1994) analyzed the dynamic user equilibrium (DUE) assignment with queues for a one-to-many / many-to-one OD pattern given the time-dependent OD volume, which means departure time from an origin or arrival time at a destination for everyone is assumed to be known. One of the most important result of the research is the decomposition scheme of the assignment based on First In First Out (FIFO) principle, which enables us to develop a convergent and tractable algorithm.

Based upon the above two types of studies, this research extends our previous analyses to include user's (commuter's) departure time choice in addition to route choice, given time constraints at the single destination. In other words, compared to the morning commute analyses, this research intends to generalize the theory to a situation in which users may pass through bottlenecks not only once but several times.

Recently, several researches on this type of dynamic equilibrium assignment models are proposed (for example, Bernstein et al.(1993), Wie et al.(1993), Ran and Boyce(1996)). Although these studies deal with a model with many-to-many OD pattern, basic properties of the model, such as existence and uniqueness of the equilibrium solution, have not been clarified at all. To achieve the full exploration of the basic properties of the model, we employ two strategies: first, we focus on the model with many-to-one OD pattern; secondly, we formulate the model by the link-node variables not path variables in the previous studies. This approach enables us to examine the model, and the results will be an important building block for the analyses of general models with many-to-many OD pattern.

Another reason for restricting our attention to the basic case of many-to-one OD pattern is due to the consideration for developing mathematically valid solution methods. As is well known, Gauss-Seidel decomposition (cyclic decomposition) is one of the valid approach to the static user equilibrium assignment problem (with many-to-many OD pattern). In this approach, a single OD equilibrium problem is solved for each OD-pair, by keeping the flows for other OD-pairs fixed. Employing this type of algorithm for solving the dynamic assignment model is a natural and promising strategy. When this strategy is applied for the dynamic assignment with many-to-many OD pattern, it is crucial to develop efficient solution methods for the sub-problem with many-to-one OD pattern. Thus, the analyses on the many-to-one OD pattern is indispensable step to develop the general models and algorithms for many-to-many OD pattern.

2. Some Definitions on a Dynamic Network

2.1. Network and OD Demand

Our model is defined on a transportation network $G[N,L]$ which has the set N of nodes with N elements, the set L of directed links with L elements and given set $\mathcal{P}$ of origin-destination(OD) node pairs with M elements. Sequential numbers from 1 to N are allocated to N nodes. A link from node i to j is denoted as link (i,j). For each of links, the cumulative arrival and departure curves are defined as follows:

$A_{ij}(t)$ = the cumulative arrivals at link (i,j) by time t,
$D_{ij}(t)$ = the cumulative departures from link (i,j) by time t.

And the derivatives of those with respect to time t are denoted as

$\lambda_{ij}(t)$ = the arrival rate at link (i,j) at time t $= dA_{ij}(t)/dt$,
$\mu_{ij}(t)$ = the departure rate from link (i,j) at time t $= dD_{ij}(t)/dt$.

The arrival rate at link (i,j) at time t, $\lambda_{ij}(t)$, is the unknown variable which must be determined so as to establish the dynamic equilibrium condition defined later. The difference between $A_{ij}(t)$ and $D_{ij}(t)$ clearly shows the number of vehicles on link (i,j) at time t, which is denoted as

$$X_{ij}(t) = \text{the number of vehicles on link } (i,j) \text{ at time } t = A_{ij}(t) - D_{ij}(t). \tag{2.1}$$

In addition, a time-dependent many-to-one OD demand is denoted as
$Q_{od}(t)$= cumulative OD demand from origin o to destination d generated at the origin by time t.

2.2. Flow Constraints

There are two kinds of flow constraints to be physically satisfied: 1) the flow conservation at nodes and 2) the First-In-First-Out queue discipline, which can be defined using variables introduced above.

(1) Flow Conservation at Nodes

The flow conservation at node k is written as follows:

$$\sum_i D_{ik}(t) - \sum_j A_{kj}(t) + Q_{kd}(t) = 0\,, \quad \forall k \in N, k \neq d \tag{2.2a}$$

where d is a destination node. The first and third terms give the cumulative number of vehicles flowing into node i by time t, while the remaining terms describe the number of vehicles leaving node i by time t. The flow conservation is also written using the time derivatives:

$$\sum_i \mu_{ik}(t) - \sum_j \lambda_{kj}(t) + dQ_{kd}(t)/dt = 0\,, \quad \forall k \in N, k \neq d \tag{2.2b}$$

(2) First In First Out Discipline

Second, under the FIFO queue discipline, a vehicle must leave link (i,j) in the same order as its order of arrival at the link. Thus, the $A_{ij}(t)$ and $D_{ij}(t)$ must be related to each other through link travel time $c_{ij}(t)$ as shown in Fig.1:

$$A_{ij}(t) = D_{ij}(t + c_{ij}(t)), \tag{2.3a}$$

where $c_{ij}(t)$ = travel time on link (i,j) for a vehicle entering the link at time t. This FIFO discipline is also described using arrival and departure rates by taking derivative of (2.3a):

$$\lambda_{ij}(t) = \mu_{ij}(t + c_{ij}(t))\,(1 + dc_{ij}(t)/dt). \tag{2.3b}$$

2.3. Link Travel Time

As in Fig.1, the FIFO discipline clearly defines link travel time $c_{ij}(t)$ such as the horizontal time difference between arrival and departure curves at arrival time t.   From (2.3a), $c_{ij}(t)$ is thus written as a function of $A_{ij}(t)$ and $D_{ij}(t)$:

$$c_{ij}(t) = D_{ij}^{-1}(A_{ij}(t)) - t. \tag{2.4}$$

Travelers are assumed to perceive $c_{ij}(t)$ as a penalty of travel, although it is possible to introduce perceived costs of travel rather than actual travel time $c_{ij}(t)$ as the conventional traffic assignment. Since the analysis is essentially the same, the link travel time is here considered as the perceived cost to eliminate further complications.

Here, the point queue concept in which a vehicle has no physical length is employed. Let μ_{ij}^* be the maximum departure rate of link (i,j) which is given, and m_{ij} be the link travel time at free flow speed. Then, departure rate $\mu_{ij}(t)$ is evaluated as follows independently of traffic condition downstream:

$$\mu_{ij}(t + c_{ij}(t)) = \begin{cases} \mu_{ij}^* & \text{if } c_{ij}(t) > m_{ij} \text{ or } \lambda_{ij}(t) > \mu_{ij}^* \\ \lambda_{ij}(t) & \text{otherwise} \end{cases} \tag{2.5}$$

If a vehicle is not delayed, it is assumed to travel on link (i,j) for free flow travel time m_{ij} which is shown as a broken line in Fig.1.   However, once link travel time $c_{ij}(t)$ gets larger than m_{ij} at time t or arrival rate $\lambda_{ij}(t)$ is larger than maximum departure rate μ_{ij}^*, departure rate $\mu_{ij}(t + c_{ij}(t))$ is restricted to μ_{ij}^* due to a queue on the link.

The (2.5) implies that if arrival curve $A_{ij}(t)$ is known by time t which means $\lambda_{ij}(t)$ is known by time t as well, $\mu_{ij}(t)$ is determined by time $t + c_{ij}(t)$ and so is $D_{ij}(t)$.   Therefore, from (2.4), $c_{ij}(t)$ basically becomes a function of only arrival curve $A_{ij}(t')$ only for $t' \leq t$ but independent of $A_{ij}(t')$ for $t' > t$.   In the deterministic queuing analysis, this result seems apparent under the point queue concept; that is, if arrival curve $A_{ij}(t)$ were known, departure curve $D_{ij}(t)$ could be drawn as the lower tangent line based on the given maximum rate μ_{ij}^* and travel time $c_{ij}(t)$ could be evaluated.

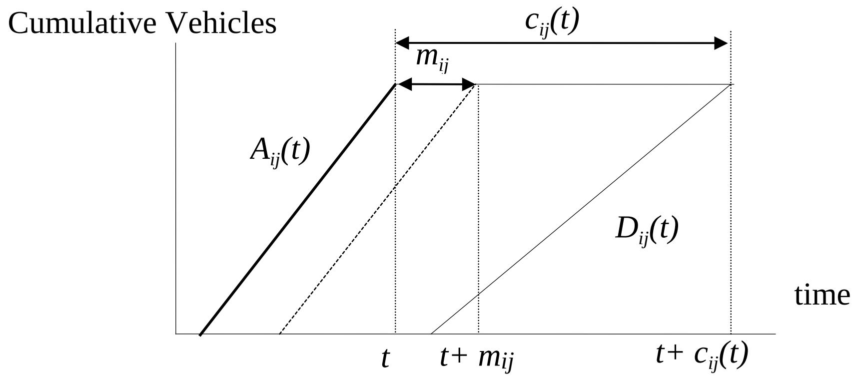

Fig. 1 Cumulative Arrival and Departure Curves on Link (i,j)

2.4. Definition of Route-Choice Equilibrium

The user equilibrium is defined as a condition where no vehicle can find a faster route than the one presently assigned to. Let $\pi_{id}(t)$ be the shortest travel time from node *i* to destination *d* at time *t*. Then, similar to the static assignment, the equilibrium condition is written such that:

$$\begin{cases} \pi_{id}(t) - \pi_{jd}(t + c_{ij}(t)) = c_{ij}(t) & \text{if a vehicle with destination } d \text{ leaving node } i \text{ at time } t \text{ uses link } (i, j), \\ \pi_{id}(t) - \pi_{jd}(t + c_{ij}(t)) \le c_{ij}(t) & \text{otherwise.} \end{cases} \tag{2.6}$$

The important difference in the above definition from the static equilibrium assignment is found in $\pi_{jd}(t + c_{ij}(t))$; that is, the shortest travel time from node *j* to *d* must be evaluated at time $t+c_{ij}(t)$ when a vehicle arrives at node *j*. Although we are considering a many-to-one OD pattern, this equilibrium definition is applicable even to a many-to-many OD pattern.

Since here we have only one destination *d*, the above definition is slightly modified by introducing $\tau_i(u)$ which is the latest arrival time at node *i* for a vehicle arriving destination *d* at time *u*. If a vehicle arrives at node *i* at time $\tau_i(u)$ and arrives at destination *d* at time *u*, clearly $\pi_{id}(t)$ is equal to $u - \tau_i(u)$. Hence, the above equilibrium condition is written using $\tau_i(u)$'s:

$$\begin{cases} \tau_j(u) - \tau_i(u) = c_{ij}(\tau_i(u)) & \text{if a vehicle arriving destination } d \text{ at time } u \text{ uses link } (i, j), \\ \tau_j(u) - \tau_i(u) \le c_{ij}(\tau_i(u)) & \text{otherwise.} \end{cases} \tag{2.7}$$

3. Basic Formulations of Dynamic Network Equilibrium Assignment

3.1. Decomposition by Arrival Time at the Single Destination

Let us first consider the order of arrivals at a node. Under the equilibrium state, a vehicle arriving at the destination earlier must arrive at any node earlier than the others arriving the same destination later than the vehicle. Thus, when the OD pattern is many-to-one, under the equilibrium state, the order of arrival at any node must be the same as the order of arrivals at the single destination *d*. And when a vehicle arrives at destination *d* at time *u*, the arrival time at node *i* must be equal to $\tau_i(u)$ (for the detailed discussion, see Kuwahara and Akamatsu(1993)).

As defined in the previous section, link travel time $c_{ij}(\tau_i(u))$ depends only on the cumulative arrival curve before time $\tau_i(u)$. Therefore, together with the above discussion on the order of arrivals at a node, it is concluded that $c_{ij}(\tau_i(u))$ depends only on route choices of those arriving the single destination before time *u*. Consequently, we can consider the assignment sequentially in the order of arrivals at single destination *d*. That is, the assignment can be decomposed with respect to arrival time *u* at destination *d* provided that the OD pattern is many-to-one.

3.2. Decomposed Representation of Flow Conservation and Link Cost Functions

In the previous section, we have concluded that the assignment problem can be decomposed regarding the arrival time at the single destination. Let us therefore consider only vehicles arriving at the destination during an interval [*u-du*, *u*] assuming that the equilibrium flow pattern of vehicles arriving at the destination before time *u-du* has been obtained (thick lines in Fig.2). First, according to the definition of the equilibrium, (2.7), if link (*i,j*) is used by vehicles arriving at destination at time *u*, $\tau_j(u)$ must be equal to $\tau_i(u) + c_{ij}(\tau_i(u))$. Thus, the FIFO discipline (2.4) becomes

$$A_{ij}(\tau_i(u)) = D_{ij}(\tau_j(u)). \qquad (3.1)$$

Applying this relationship to (2.2a), we obtain the flow conservation at node *k* in the following way eliminating D_{ij}:

$$\sum_i A_{ik}(\tau_i(u)) - \sum_j A_{kj}(\tau_k(u)) + Q_{kd}(\tau_k(u)) = 0, \quad \forall k \in N, k \neq d. \qquad (3.2a)$$

This flow conservation is also described using the arrival flow rate by taking derivatives with respect to arrival time *u*:

$$\sum_i y_{ik}(u) - \sum_j y_{kj}(u) + q_{kd}(u) = 0, \quad \forall k \in N, k \neq d \qquad (3.2b)$$

where $y_{ik}(u) \equiv A_{ik}(\tau_i(u))/du$, $q_{kd}(u) \equiv Q_{kd}(\tau_k(u))/du$.

And considering time interval [*u-du*, *u*], travel time of link (*i,j*) for a user entering into the link at time $\tau_i(u)$ is now described as a function of $y_{ij}(u)$ and $\tau_i(u)$ as below:

$$c_{ij}(\tau_i(u)) = \text{Max.}[c_{ij}(\tau_i(u-du)) + \{X_{ij}(\tau_i(u)) - X_{ij}(\tau_i(u-du))\}/\mu^*_{ij},\ m_{ij}]$$

$$= \text{Max.}[c_{ij}(\tau_i(u-du)) + y_{ij}(u) \cdot du/\mu^*_{ij} - \{\tau_i(u) - \tau_i(u-du)\},\ m_{ij}]. \qquad (3.3)$$

Since unknowns are only $y_{ij}(u)$ and $\tau_i(u)$ in the above equation but $c_{ij}(\tau_i(u-du))$ and $\tau_i(u-du)$ have been evaluated from thick lines in Fig.2, the travel time can be simply written as

$$c_{ij}(\tau_i(u)) = \text{Max.}[\alpha_{ij} y_{ij}(u) + \beta_{ij} - \tau_i(u)\ ,\ m_{ij}] \qquad (3.4a)$$

where $\alpha_{ij} \equiv du/\mu^*_{ij}$, $\beta_{ij} = c_{ij}(\tau_i(u-du)) + \tau_i(u-du)$. (3.4b)

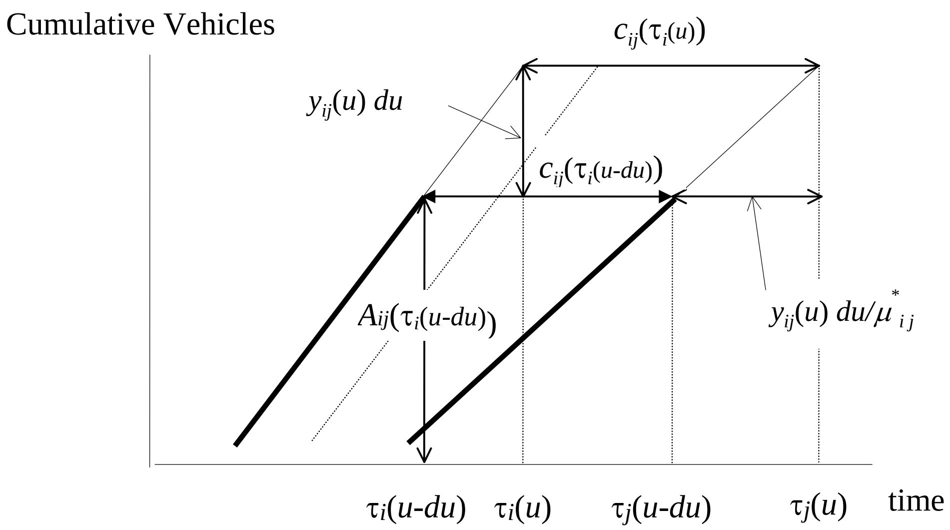

Fig.2 Travel Time on Link (*i,j*) for a vehicle arriving at destination at time *u*

3.3 Decomposed Representation of Route-Choice Equilibrium

The equilibrium condition (2.7) is equivalently written as the following complementarity condition:

$$\begin{cases} y_{ij}(u)\cdot(c_{ij}(\tau_i(u))+\tau_i(u)-\tau_j(u))=0 \\ \qquad c_{ij}(\tau_i(u))+\tau_i(u)-\tau_j(u)\ \geq 0, \quad y_{ij}(u)\ \geq 0 \end{cases} \qquad \forall(i,j)\in L \qquad (3.5)$$

$$\sum_i y_{ik}(u)-\sum_j y_{kj}(u)+q_{kd}(u)=0 \ . \qquad \forall k\in N, k\neq d \qquad (3.2b)$$

$$c_{ij}(\tau_i(u))=\text{Max.}[\alpha_{ij}y_{ij}(u)+\beta_{ij}-\tau_i(u)\ ,\ m_{ij}] \qquad (3.4a)$$

In this formulation, unknowns are $y_{ij}(u)$'s of all L links, and $\tau_i(u)$'s at all nodes except destination d (or OD demand $q_{od}(u)$'s). Note that the link travel time $c_{ij}(\tau_i(u))$ depends on not only $y_{ij}(u)$ but also $\tau_i(u)$ while the link cost in the static assignment is a function of only link flows which correspond to $y_{ij}(u)$'s here.

3.4 Decomposed Representation of OD Demand

In this paper we explore the properties of the dynamic assignment for two kind of OD demand models: first model is the simplest one based on the assumption of deterministic and homogeneous user behavior; second model is the LOGIT type stochastic one that is a generalization of the first model.

(1) Deterministic Demand Model

Let us now define the first OD demand model. We begin with the following assumption on the users' behavior:

1) The users choose their departure time so as to minimize their experienced disutility.
2) The users are homogeneous in the sense that they have the same disutility function.
3) The disutility function consists of their experienced travel time and "schedule cost":

$$V_{od}(u,\ v)=(u-\tau_o(u))+\psi_{od}(v-u)\,, \qquad (3.6)$$

where V_{od} (u, v) denotes the disutility of a traveler with work starting time v who departs from origin o and arrives at destination d at time u, $\psi_{od}(v-u)$ stands for the convex and non-negative schedule cost function, v - u is the schedule delay for a traveler with work starting time of v.

Under these assumption, the equilibrium OD demand is given by the solution of the following complementarity conditions and the flow conservations:

$$\begin{cases} q_{od}(u)\cdot\{V_{od}(u,v)-\rho_{od}\}=0 \\ \qquad V_{od}(u,v)-\rho_{od}\ \geq 0, \quad q_{od}(u)\ \geq 0 \end{cases} \quad \forall od\in \mathscr{P},\ u\in[0,T] \qquad (3.7)$$

$$\int_0^T q_{od}(u)du=Q_{od}(T) \qquad \forall od\in \mathscr{P} \qquad (3.8)$$

Condition (3.7) means that the disutility for the arrival time u, V_{od} (u, v), is minimized to a certain equilibrium level, ρ_{od}, if the OD demand for the arrival time u, $q_{od}(u)$, is positive; the disutility for u is greater than ρ_{od} (i.e. $V_{od}(u,v)>\rho_{od}$) if the OD demand is zero.

(2) Stochastic Demand Model

The second OD demand model is based upon the random utility theory. The users are assumed to choose their arrival time so as to minimize their perceived disutility of the OD pair:

$$\tilde{U}_{od}(u,v) = V_{od}(u,v) + \tilde{\varepsilon} \tag{3.9}$$

where $\tilde{U}_{od}(u,v)$ denotes the perceived (random) disutility of a traveler with work starting time t_w who departs from origin *o* and arrives at destination *d* at time *u*, V_{od} (*u*, *v*) is defined in (3.6) and $\tilde{\varepsilon}$ is a error term. When the random error term follows the i.i.d. Gumbell distribution, the probability density that a traveler with work starting time *v* who departs from origin *o* arrives at destination *d* at time *u* is given by

$$p_{od}(u,v) = \frac{\exp[-\theta V_{od}(u, v)]}{\int_{\underline{u}}^{\bar{u}} \exp[-\theta V_{od}(u', v)]du'} \tag{3.10}$$

where θ is a distribution parameter of error term. In addition, we assume that the work start time , *v*, are distributed among users, and the distribution for each OD pair is given as the function $w_{od}(\cdot)$. Then, the OD demand for the destination arrival time *d* is

$$q_{od}(u) = Q_{od} \int_{\underline{w}}^{\bar{w}} w_{od}(v) p_{od}(u,v) dv, \tag{3.11}$$

where Q_{od} denote the given total OD demand from origin o to destination d.

Since substantial amount of work on departure time choice for morning commute has been reported, it is advantageous to compare the definition of the demand model above with those previous studies. The previous analyses mostly employ the deterministic departure time choice as in (1) and hence they correspond to our second demand model with $\theta \to \infty$. Also, they are based upon the special property of the First In First Work (FIFW) discipline, which means that, in the equilibrium, the order of arrivals at any node is the same as the order of work starting times at offices (or equivalently same as the desired departure times from bottlenecks). For the single bottleneck analyses, Smith(1984) and Daganzo(1985) proved the FIFW discipline provided that the schedule cost function $\psi_{od}(v-u)$ is convex in schedule delay *v-u*. Kuwahara and Newell(1987) then showed that the FIFW discipline is valid when commuters pass through bottlenecks only once even for a many-to-one OD pattern, and decomposed their problem including route choice with respect to work starting times *v*'s. Even in our demand model with $\theta \to \infty$, once the FIFW discipline is guaranteed, the problem can be decomposed with respect to *v*'s. However, the FIFW discipline cannot be always established for a general network with a many-to-one OD pattern where travelers may get through two or more bottlenecks taking various routes to the single destination.

4. Alternative Formulations - VIP, NCP and FPP

In this section, we convert the DUE assignment described in the previous section into various formulations that are more convenient for the mathematical analyses: Variational Inequality Problems (VIP), Non-linear Complementarity Problems (NCP) and Fixed Point Problems (FPP). In each formulation, we classify the models into two categories based on the demand function shown in section 3.4 and the corresponding formulations are derived in parallel.

Recently, Smith(1993) and Friesz et al.(1993) also formulated the DUE assignment as a VIP, which we refer to it as VIP-DUE-path since the formulation is based on *path variables*. Unlike the VIP-DUE-path, our formulation shown below is based on *node / link variables* decomposed with respect to arrival times at a destination. This strategy gives us several advantages over VIP-DUE-path. First, it is easy to analyze the mathematical properties of the problem since the mapping appearing in our formulation is very simple comparing with that in VIP-DUE-path. Second, this formulation enables us to define a easily computable *merit function*, which is a useful tool for developing efficient and convergent algorithms. The precise demonstrations of these advantages of our formulations will be presented in the following sections.

4.1 Variational Inequality Formulations

As we have seen in 3.1, we can decompose the DUE assignment model by the arrival time at a destination, u, when the OD demand is fixed. The OD demand $q_{od}(u)$, however, cannot be determined by considering only time interval of $[u\text{-}du, u]$ but we have to consider the whole study period $[\overline{u}, \underline{u}]$ because of the integration in the denominator of (3.10). Thus, we must consider the equilibrium conditions for a whole time period simultaneously.

For the convenience of the analysis in the later sections, we formulate the DUE model by discretizing the arrival time at a destination: we divide an underlying time period into a finite number, K, of intervals and the set of the intervals is denoted as $\mathcal{T}$. We also use a superscript u for the arrival time at a destination in the set $\mathcal{T}$, and the following vector notation is used:

$\mathbf{c}^u=(.., c_{ij}(u) ,..)^T \in \boldsymbol{R}^{\mathrm{L}}$, $\mathbf{c}=(\mathbf{c}^0...,\mathbf{c}^u,...\mathbf{c}^{\mathrm{K}})^T \in \boldsymbol{R}^{\mathrm{L\cdot K}}$, $\mathbf{y}^u=(.., y_{ij}(u) ,..)^T \in \boldsymbol{R}^{\mathrm{L}}$, $\mathbf{y}=(\mathbf{y}^0,...,\mathbf{y}^u,...,\mathbf{y}^{\mathrm{K}})^T \in \boldsymbol{R}^{\mathrm{L\cdot K}}$,

$\boldsymbol{\tau}^u=(.., \tau_o(u) ,..)^T \in \boldsymbol{R}^{\mathrm{N}}$, $\boldsymbol{\tau}=(\boldsymbol{\tau}^0,...,\boldsymbol{\tau}^u,...,\boldsymbol{\tau}^{\mathrm{K}})^T \in \boldsymbol{R}^{\mathrm{N\cdot K}}$, $\mathbf{q}^u = (.., q_{od}(u) ,..)^T \in \boldsymbol{R}^{\mathrm{M}}$, $\mathbf{q} = (\mathbf{q}^0,..., \mathbf{q}^u,...,\mathbf{q}^{\mathrm{K}})^T \in \boldsymbol{R}^{\mathrm{M\cdot K}}$

$\boldsymbol{\Psi}^u=(.., \psi_{od}(u) ,..)^T \in \boldsymbol{R}^{\mathrm{M}}$, $\boldsymbol{\Psi}=(\boldsymbol{\Psi}^0,...,\boldsymbol{\Psi}^u,...,\boldsymbol{\Psi}^{\mathrm{K}})^T \in \boldsymbol{R}^{\mathrm{M\cdot K}}$, $\boldsymbol{\rho} = (.., \rho_{od} ,..)^T \in \boldsymbol{R}^{\mathrm{M}}$, $\mathbf{Q} = (.., Q_{od} ,..)^T \in \boldsymbol{R}^{\mathrm{M}}$.

1) Deterministic Demand Function Case

In the discrete time system, the path choice DUE conditions (3.5) can be represented as

$$\mathbf{y}^u \cdot (\mathbf{c}^u - \mathbf{A}^T \boldsymbol{\tau}^u) = 0, \quad \mathbf{c}^u - \mathbf{A}^T \boldsymbol{\tau}^u \geqq \mathbf{0}, \quad \mathbf{y}^u \geqq \mathbf{0} \quad \forall u \in \mathcal{T} \tag{4.1a}$$

or

$$\mathbf{y} \cdot (\mathbf{c} - \mathbf{A}_{\mathrm{K}}^T \boldsymbol{\tau}) = 0, \quad \mathbf{c} - \mathbf{A}^T \boldsymbol{\tau} \geqq \mathbf{0}, \quad \mathbf{y} \geqq \mathbf{0} \tag{4.1b}$$

where A and A_K denote a node-link incidence matrix, and the block diagonal matrix with K diagonal blocks all equal to A, respectively. The flow conservation (3.2b) is represented as

$$\mathbf{q}^u - \mathbf{A}\,\mathbf{y}^u = \mathbf{0} \qquad \forall u \in \mathcal{T} \tag{4.2a}$$

or
$$\mathbf{q} - \mathrm{A_K}\,\mathbf{y} = \mathbf{0} \tag{4.2b}$$

Similarly, the deterministic OD demand conditions (3.7) can be written as

$$\mathbf{q}\cdot(\mathbf{u} - \tau + \mathbf{\Psi} - \mathrm{D}^T\rho) = 0,\quad \mathbf{u} - \tau + \mathbf{\Psi} - \mathrm{D}^T\rho \geqq \mathbf{0},\quad \mathbf{q} \geqq \mathbf{0} \tag{4.3}$$

where D is the M×(M•K) block diagonal matrix with K diagonal blocks all equal to [1,...1] = the "arrival time-OD pair" incidence matrix. The flow conservation (3.8) is also represented as

$$\mathrm{D}\,\mathbf{q} - \mathbf{Q} = \mathbf{0}. \tag{4.4}$$

Now, define $\mathbf{X} \in K_1 = \boldsymbol{R}_+^{\mathrm{L\times K}} \times \boldsymbol{R}^{\mathrm{N\times K}} \times \boldsymbol{R}_+^{\mathrm{M\times K}} \times \boldsymbol{R}^{\mathrm{M}}$ and mapping $\mathbf{F}$: $K_1 \rightarrow K_1$ as follows:

$$\mathbf{X} = \begin{bmatrix}\mathbf{y}\\ \tau\\ \mathbf{q}\\ \rho\end{bmatrix},\quad \mathbf{F}(\mathbf{X}) = \begin{bmatrix} 0 & \mathrm{A_K}^T & 0 & 0\\ -\mathrm{A_K} & 0 & \mathrm{I} & 0\\ 0 & -\mathrm{I} & 0 & -\mathrm{D}^T\\ 0 & 0 & \mathrm{D} & 0\end{bmatrix}\begin{bmatrix}\mathbf{y}\\ \tau\\ \mathbf{q}\\ \rho\end{bmatrix} + \begin{bmatrix}\mathbf{c}(\mathbf{y},\tau)\\ \mathbf{0}\\ \mathbf{u}-\psi\\ -\mathbf{Q}\end{bmatrix} \tag{4.5}$$

Then, we can state the VIP formulation of the DUE assignment with deterministic OD demand.

Theorem 4.1. *The vector* $\mathbf{X}^* \in K_1$ *is a solution of simultaneous DUE assignment with deterministic OD demand if and only if it satisfies the following VIP,* $VI(K_1, \mathbf{F})$:

$$\textit{Find}\quad \mathbf{X}^* \in K_1 \;\; \textit{such that}\quad \mathbf{F}(\mathbf{X})\cdot(\mathbf{X} - \mathbf{X}^*) \geqq 0 \qquad \forall\, \mathbf{X} \in K_1. \tag{4.6}$$

Proof: In expanded form, the VI (4.1) is represented as

$$(\mathbf{c}(\mathbf{y}^*,\tau^*) + \mathrm{A_K}^T\boldsymbol{\tau}^*)\cdot(\mathbf{y} - \mathbf{y}^*) + (\mathbf{q}^* - \mathrm{A_K}\,\mathbf{y}^*)\cdot(\tau - \tau^*)$$
$$+ (\tau^* + \mathrm{u} - \mathbf{\Psi} - \mathrm{D}^T\rho^*)\cdot(\mathbf{q} - \mathbf{q}^*) + (\mathrm{D}\,\mathbf{q}^* - \mathbf{Q})\cdot(\rho - \rho^*) \geqq 0 \quad \forall(\mathbf{y}, \tau, \mathbf{q}, \rho) \in K_1 \tag{4.7}$$

Assume that $\mathbf{X}^*$ satisfies the simultaneous DUE conditions (4.1)-(4.3). We will show that $\mathbf{X}^*$ must satisfies VI (4.4). It follows that (4.1) implies

$$\{c_{ij}(y_{ij}^*(u), \tau_i^*(u)) + \tau_i^*(u) - \tau_j^*(u)\}(y_{ij}(u) - y_{ij}^*(u)) \geq 0 \quad \forall y_{ij}(u) \in R_+, \forall (i,j) \in L, \forall u \in \mathcal{T}, \tag{4.8a}$$

Similarly, (4.2), (4.3) and (4.4) imply

$$\{\sum_i y_{ik}^*(u) - \sum_j y_{kj}^*(u) + q_{kd}(\tau_k^*(u))\}(\tau_k(u) - \tau_k^*(u)) = 0 \quad \forall \tau_k(u) \in R, \forall k \in N, \forall u \in \mathcal{T}, \tag{4.8b}$$

$$\{V_{od}(u, t_w) - \rho_{od}^*\}(q_{od}(u) - q_{od}^*(u)) \geq 0 \quad \forall q_{od}(u) \in R_+, \forall (o,d) \in \mathcal{P}, \;\; \forall u \in \mathcal{T}, \tag{4.8c}$$

$$\{\sum_u q_{od}^*(u) - Q_{od}\}(\rho_{od} - \rho_{od}^*) = 0 \quad \forall \rho_{od} \in R, \quad \forall od \in \mathcal{P}, \tag{4.8d}$$

respectively. Summing (4.8a),(4.8b),(4.8c) and (4.8d) over $\{(i,j) \in L,\ u \in \mathcal{T}\}$, $\{k \in N,\ u \in \mathcal{T}\}$, $\{(o,d) \in \mathcal{P},\ u \in \mathcal{T}\}$ and $\{(o,d) \in \mathcal{P}\}$, respectively, we obtains VI (4.6).

Next, let us consider the converse. Assume that $\mathbf{X}^*$ satisfies (4.6). We will show that $\mathbf{X}^*$ also satisfies DUE conditions (4.1)-(4.4). Set $\boldsymbol{\tau} = \boldsymbol{\tau}^*$, $\mathbf{q}=\mathbf{q}^*$, $\rho=\rho^*$ and $y_{kl}(u) = y_{kl}^*(u)$ for all $\{(k,l) \neq (i,j),\ u \neq v\}$ where (i,j) and v are an arbitrary fixed link in L, and an arbitrary arrival time in $\mathcal{T}$, respectively. Then, (4.6) reduces to (4.8a), which implies that (4.1) must hold. Similarly, set $\mathbf{y} = \mathbf{y}^*$, $\mathbf{q}=\mathbf{q}^*$, $\boldsymbol{\rho}=\boldsymbol{\rho}^*$ and $\tau_l(u)=\tau_l^*(u)$ for all $\{l \neq k,\ u \neq v\}$ where k and v are an arbitrary fixed node in N, and an arbitrary arrival time in $\mathcal{T}$. Then, (4.6) reduces to (4.8b), which implies that (4.2) must hold. In the almost same manner, we can obtain (4.8c) and (4.8d) from (4.6), which imply that (4.3) and (4.4) hold. This completes the proof.□

To compare our model with the previous studies, we will show the equivalent path-based formulation below, where the following notation are used:

$\mathcal{R}_{od}$: a set of routes between origin-destination pair od with H_{od} elements, $\mathcal{R} \equiv \bigcap_{od} \mathcal{R}_{od}$, $\mathrm{H} \equiv \sum_{od} \mathrm{H}_{od}$,

$F_r^{od}(u)$: Number of vehicles on rth path for OD pair od that arrives at destination by time u,

$f_r^{od}(u) = d F_r^{od}(u) / du$, $\mathbf{f}^u = (.., f_r^{od}(u),..)^T \in \boldsymbol{R}^{\boldsymbol{H}}$ $\forall u \in \mathcal{T}$, $\mathbf{f} = (.., \mathbf{f}^u,..)^T \in \boldsymbol{R}^{\boldsymbol{K \cdot H}}$,

$\gamma_{od}(u)$: equilibrium OD travel time for a user with OD pair od who arrives at destination at time u,

$\boldsymbol{\gamma}^u = (.., \gamma_{od}(u),..)^T \in \boldsymbol{R}^{\boldsymbol{M}}$ $\forall u \in \mathcal{T}$, $\boldsymbol{\gamma} = (.., \boldsymbol{\gamma}^u,..)^T \in \boldsymbol{R}^{\boldsymbol{K \cdot M}}$,

$C_r^{od}(u)$: travel time for a user who uses rth path for OD pair od and arrives at destination at time u,

$\mathbf{C}^u = (.., C_r^{od}(u),..)^T \in \boldsymbol{R}^{\boldsymbol{H}}$ $\forall u \in \mathcal{T}$, $\mathbf{C} = (.., \mathbf{C}^u,..)^T \in \boldsymbol{R}^{\boldsymbol{K \cdot H}}$.

The shortest path conditions corresponding to (4.1) are given by

$$\mathbf{f}^u \cdot (\mathbf{C}^u - \mathrm{B}^T \boldsymbol{\gamma}^u) = 0, \quad \mathbf{C}^u - \mathrm{B}^T \boldsymbol{\gamma}^u \geqq \mathbf{0}, \quad \mathbf{f}^u \geqq \mathbf{0} \qquad \forall u \in \mathcal{T} \tag{4.9a}$$

$$or \qquad \mathbf{f} \cdot (\mathbf{C} - \mathrm{B_K}^T \boldsymbol{\gamma}) = 0, \quad \mathbf{C} - \mathrm{B_K}^T \boldsymbol{\gamma} \geqq \mathbf{0}, \quad \mathbf{f} \geqq \mathbf{0} \tag{4.9b}$$

where B is a “route - OD pair” incidence matrix ($\mathrm{M} \times \mathrm{H}_{od}$ matrix), $\mathrm{B_K}$ is the $(\mathrm{M} \cdot \mathrm{K}) \times (\mathrm{H} \cdot \mathrm{K})$ block diagonal matrix with K diagonal blocks all equal to B. Using these matrices, the flow conservation corresponding to (4.2) are represented as

$$\mathrm{B}\, \mathbf{f}^u - \mathbf{q}^u = \mathbf{0} \qquad \forall u \in \mathcal{T} \tag{4.10a}$$

$$or \qquad \mathrm{B_K}\, \mathbf{f} - \mathbf{q} = \mathbf{0} \tag{4.10b}$$

The OD demand condition can be represented by almost same manner as in (4.3) and (4.4). Thus, the equivalent VI problem corresponding to (4.6) is given as follows:

$$\textit{Find} \quad \mathbf{X}^* \in K_{1\mathrm{P}} \quad \textit{such that} \quad \mathbf{F}(\mathbf{X}) \cdot (\mathbf{X} - \mathbf{X}^*) \geqq 0 \quad \forall\, \mathbf{X} \in K_{1\mathrm{P}} . \tag{4.11a}$$

where $\mathbf{X} \in K_{1\mathrm{P}} = \boldsymbol{R}_+^{\mathrm{H \times K}} \times \boldsymbol{R}^{\mathrm{M \times K}} \times \boldsymbol{R}_+^{\mathrm{M \times K}} \times \boldsymbol{R}^{\mathrm{M}}$ and mapping $\mathbf{F}: K_{1\mathrm{P}} \to K_{1\mathrm{P}}$ are defined as

$$\mathbf{X} = \begin{bmatrix} \mathbf{f} \\ \boldsymbol{\gamma} \\ \mathbf{q} \\ \boldsymbol{\rho} \end{bmatrix}, \quad \mathbf{F}(\mathbf{X}) = \begin{bmatrix} 0 & -\mathrm{B_K}^T & 0 & 0 \\ \mathrm{B_K} & 0 & -\mathrm{I} & 0 \\ 0 & \mathrm{I} & 0 & -\mathrm{D}^T \\ 0 & 0 & \mathrm{D} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \boldsymbol{\gamma} \\ \mathbf{q} \\ \boldsymbol{\rho} \end{bmatrix} + \begin{bmatrix} \mathbf{C}(\mathbf{f}) \\ \mathbf{0} \\ \boldsymbol{\Psi} \\ \mathbf{Q} \end{bmatrix} \tag{4.11b}$$

The VI problem above is somewhat redundant. In fact, the shortest path condition (4.9) and the minimum disutility condition (4.3) can be combined as follows:

$$\mathbf{f} \cdot (\mathbf{C} + \mathrm{B_K}^T \boldsymbol{\Psi} - \mathrm{B_K}^T \mathrm{D}^T \boldsymbol{\rho}) = 0, \quad \mathbf{C} + \mathrm{B_K}^T \boldsymbol{\Psi} - \mathrm{B_K}^T \mathrm{D}^T \boldsymbol{\rho} \geqq \mathbf{0}, \quad \mathbf{f} \geqq \mathbf{0} \tag{4.12}$$

The “simultaneous minimization” of disutility in (4.12) is equivalent to the “double stage minimization” in (4.11), since the following holds:

$$\min_{r,u}.\{V_r^{od}(u)\} = \min_u\{\min_r.\{V_r^{od}(u)\}\}$$
$$= \min_u\{\gamma_{od}(u) + \Psi_{od}(u)\} = \min_u\{V_{od}(u)\}\,.$$

In this formulation, the required flow conservation also reduces to

$$\mathrm{D}\,\mathrm{B}_{\mathrm{K}}\,\mathbf{f} - \mathbf{Q} = \mathbf{0}\,. \qquad (4.13)$$

Thus, we have the following “reduced” VI problem that is equivalent with (4.11):

Find $\mathbf{X}^{*}\in K_{1\mathrm{P}-1}=\boldsymbol{R}^{\mathrm{H}\times\mathrm{K}}\times\boldsymbol{R}^{\mathrm{M}\times\mathrm{K}}$ *such that* $\mathbf{F}(\mathbf{X})\cdot(\mathbf{X}-\mathbf{X}^{*})\geqq 0 \quad \forall\mathbf{X}\in K_{1\mathrm{P}-1}$. (4.14a)

where $\mathbf{X}\in K_{1\mathrm{P}}$ and mapping $\mathbf{F}{:}K_{1\mathrm{P}-1}\rightarrow K_{1\mathrm{P}-1}$ are defined as

$$\mathbf{X}=\begin{bmatrix}\mathbf{f}\\ \boldsymbol{\rho}\end{bmatrix},\quad \mathbf{F}(\mathbf{X})=\begin{bmatrix}0 & -\mathrm{B_K}^{T}\mathrm{D}^{T}\\ \mathrm{D}\,\mathrm{B_K} & 0\end{bmatrix}\begin{bmatrix}\mathbf{f}\\ \boldsymbol{\rho}\end{bmatrix}+\begin{bmatrix}\mathbf{C}(\mathbf{f})+\mathrm{B_K}^{T}\boldsymbol{\Psi}\\ \mathbf{Q}\end{bmatrix}. \qquad (4.14b)$$

Furthermore, it is easily seen that the cost variable $\boldsymbol{\rho}^{*}$ in the “Primal-Dual” VIP (4.14) can be eliminated by “dropping” the term in F(**X**) corresponding to (4.13) into constraint set. That is, VIP (4.14) yields the following “Primal” type VIP:

Find $\mathbf{f}^{*}\in \mathrm{K}_{1\mathrm{P}-2}$ *such that* $\{\mathbf{C}(\mathbf{f}^{*}) + \mathrm{B_K}\,\boldsymbol{\Psi}\}\cdot(\mathbf{f}-\mathbf{f}^{*})\geqq 0 \quad \forall\mathbf{f}\in\mathrm{K}_{1\mathrm{P}-2}$ (4.15)

where $\mathrm{K}_{1\mathrm{P}-2}=\{\mathbf{f}\mid \mathbf{Q}=\mathrm{D}\,\mathrm{B_K}\,\mathbf{f},\quad \mathbf{f}\geqq\mathbf{0}\}$

This is the problem proposed by Friesz et al.(1993) and Bernstein et al.(1993). Although the problem (4.15) is seemingly simple, it is difficult to analyze the basic properties. The reason is that **C**(**f**) in this formulation is very complex mapping of various path flows**,** which can not be represented analytically. On the other hand, the VIP (4.6) derived from link-node formulation is easy to analyze, since the mapping **F**(**X**)consists of relatively simple link cost **c(***u***)** that is expressed as an analytical function.

2) Stochastic Demand Function Case

In addition to the (destination) arrival time u in the set $\mathscr{T}$, we divide the work start time into a finite number, J, of intervals and the set of the intervals is denoted as $\mathscr{S}$. We also use a superscript v for the work start time in the set $\mathscr{S}$. For the set $\mathscr{S}$, the work start time distribution satisfys

$$\sum_{v\in\mathscr{S}} w_{od}(v)=Q_{od} \qquad \forall od\in\mathscr{P}$$

or $\mathrm{E}\,\mathbf{w} = \mathbf{Q}$, (4.16)

where E is the “work start time-od pair” incidence matrix (which is the M×(M・J) block diagonal matrix with J diagonal blocks all equal to [1...1]) .

Given the work start time distribution **w** for the discrete time system, the Logit type OD demand (3.10) is represented as

$$q_{od}^{u,v} = w_{od}^{v} \cdot p_{od}^{u,v} = w_{od}^{v} \frac{\exp[-\theta V_{od}^{u,v}]}{\sum_{u' \in \mathcal{J}} \exp[-\theta V_{od}^{u',v}]}, \quad (4.17)$$

where $q_{od}^{u,v}$ is the OD demand with work starting time v that departs from origin o arrives at destination d at time u. This can be alternatively represented as the following complementarity conditions:

$$\begin{cases} q_{od}^{u,v} \cdot \{\frac{1}{\theta} \ln q_{od}^{u,v} + V_{od}^{u,v} - \rho_{od}^{v}\} = 0 \\ \frac{1}{\theta} \ln q_{od}^{u,v} + V_{od}^{u,v} - \rho_{od}^{v} \geq 0, \quad q_{od}^{u,v} \geq 0 \end{cases} \quad \forall u \in \mathcal{J}, \ \forall v \in \mathcal{S} \quad (4.18a)$$

$$\text{and} \ \sum_{u \in \mathcal{J}} q_{od}^{u,v} = w_{od}^{v}. \quad \forall v \in \mathcal{S} \quad (4.18b)$$

Although the variable $\rho_{od}^{u,v}$ in (4.17) is not represented as an explicit function of $V_{od}^{u',v}$, combining (4.18a) and (4.18b) yields

$$\rho_{od}^{u,v} = -\frac{1}{\theta}\{\ln \sum_{u' \in \mathcal{J}} \exp[-\theta V_{od}^{u',v}] + \ln w_{od}^{v}\},$$

and the equivalence between (4.17) and (4.18) can be easily verified.

Expressing (4.18a) and (4.18b) in a vector-matrix form, we have

$$\mathbf{q}^{v} \cdot (\frac{1}{\theta} \mathbf{ln}\, \mathbf{q}^{v} + \mathbf{V}^{v} - \mathrm{D}^{T} \boldsymbol{\rho}^{v}) = 0, \ \frac{1}{\theta} \mathbf{ln}\, \mathbf{q}^{v} + \mathbf{V}^{v} - \mathrm{D}^{T} \boldsymbol{\rho}^{v} \geq \mathbf{0}, \ \mathbf{q}^{v} \geq \mathbf{0} \ \ \forall v \in \mathcal{S} \quad (4.19a)$$

$$\mathrm{D}\, \mathbf{q}^{v} - \mathbf{w}^{v} = \mathbf{0} \qquad \forall v \in \mathcal{S}. \quad (4.19b)$$

or equivalently,

$$\mathbf{q} \cdot (\frac{1}{\theta} \mathbf{ln}\, \mathbf{q} + \mathrm{E_K}^{T} (\mathbf{u} - \boldsymbol{\tau}) + \boldsymbol{\psi} - \mathrm{D_J}^{T} \boldsymbol{\rho}) = 0, \ \frac{1}{\theta} \mathbf{ln}\, \mathbf{q} + \mathrm{E_K}^{T} (\mathbf{u} - \boldsymbol{\tau}) + \boldsymbol{\psi} - \mathrm{D_J}^{T} \boldsymbol{\rho} \geq \mathbf{0}, \ \mathbf{q} \geq \mathbf{0} \quad (4.20a)$$

$$\mathrm{D_J} \mathbf{q} - \mathbf{w} = \mathbf{0} \quad (4.20b)$$

where $\mathrm{D_J}$ is the (M•J)×(M•K•J) block diagonal matrix with J diagonal blocks all equal to D,
$\mathrm{E_K}$ is the (M•K)×(M•K•J) block diagonal matrix with K diagonal blocks all equal to E,
$\mathbf{ln}\, \mathbf{z}$ denotes the vector with components $\ln z_i$, $i = 1,2,..n$, for any vector $\mathbf{z} \in \boldsymbol{R}^{n}$.

Now, define $\mathbf{X} \in K_2 = \boldsymbol{R}_{+}^{\mathrm{L \times K}} \times \boldsymbol{R}^{\mathrm{N \times K}} \times \boldsymbol{R}_{+}^{\mathrm{M \times J \times K}} \times \boldsymbol{R}^{\mathrm{M \times J}}$ and mapping $\mathbf{F}: K_2 \rightarrow K_2$ as follows:

$$\mathbf{X} = \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\tau} \\ \mathbf{q} \\ \boldsymbol{\rho} \end{bmatrix}, \quad \mathbf{F}(\mathbf{X}) = \begin{bmatrix} 0 & \mathrm{A_K}^{T} & 0 & 0 \\ -\mathrm{A_K} & 0 & \mathrm{E_K} & 0 \\ 0 & -\mathrm{E_K}^{T} & 0 & -\mathrm{D_J}^{T} \\ 0 & 0 & \mathrm{D_J} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\tau} \\ \mathbf{q} \\ \boldsymbol{\rho} \end{bmatrix} + \begin{bmatrix} \mathbf{c}(\mathbf{y}, \boldsymbol{\tau}) \\ \mathbf{0} \\ \mathrm{E_K}^{T} \mathbf{u} - \boldsymbol{\Psi} + (1/\theta)\, \mathbf{ln}\, \mathbf{q} \\ -\mathbf{w} \end{bmatrix} \quad (4.21)$$

Then, we can state the VIP formulation of the DUE assignment with stochastic OD demand.

Theorem 4.2A. *The vector* $\mathbf{X}^* \in K_2$ *is a solution of simultaneous DUE assignment with LOGIT type stochastic OD demand for a time period* $\mathcal{T}$ *if and only if it satisfies the following VIP,* $VI(K_2, \mathbf{F})$:

$$\textit{Find a vector } \mathbf{X}^* \in K_2 \textit{ such that } \mathbf{F}(\mathbf{X}^*)\cdot(\mathbf{X}-\mathbf{X}^*) \geq \mathbf{0} \qquad \forall\ \mathbf{X} \in K_2 \tag{4.22}$$

The formulation above is restricted to the LOGIT type OD demand function. For the general demand function $\mathbf{q}(\boldsymbol{\tau})$, the VI representation is stated as follows:

Theorem 4.2B. *The vector* $\mathbf{x}^* \in K_{2B} = \boldsymbol{R}_+^{\mathrm{L}\times\mathrm{K}} \times \boldsymbol{R}^{\mathrm{N}\times\mathrm{K}}$ *is a solution of simultaneous DUE assignment with general OD demand for a time period* $\mathcal{T}$ *if and only if it satisfies the following VIP,* $VI(K_{2B}, \mathbf{F})$ *:*

$$\textit{Find a vector } \mathbf{x}^* \in K_{2B} \textit{ such that } \mathbf{F}(\mathbf{x}^*)\cdot(\mathbf{x}-\mathbf{x}^*) \geq \mathbf{0} \qquad \forall \mathbf{x} \in K_{2B} \tag{4.23}$$

where a vector $\mathbf{x} \in K_{2B}$ *and a mapping* $\mathbf{F}(\mathbf{x})$: $K_{2B} \rightarrow K_{2B}$ *are defined by*

$$\mathbf{x} \equiv \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\tau} \end{bmatrix}, \quad \mathbf{F}(\mathbf{x}) \equiv \begin{bmatrix} \mathbf{c}(\mathbf{y},\boldsymbol{\tau}) \\ \mathbf{q}(\boldsymbol{\tau}) \end{bmatrix} + \begin{bmatrix} 0 & \mathrm{A_K}^T \\ -\mathrm{A_K} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\tau} \end{bmatrix}. \tag{4.24}$$

A few remarks are in order for the VIP above. The structure of the formulation (4.23) is very similar to the (multi-commodity) spatial price equilibrium model (see, for example, Pang(1984), Friesz et al.(1983) etc.) or (naturally) a link/node formulation of the static Wardrop equilibrium assignment. There is, however, a clear difference between the VIPs for the static network equilibrium problems and that for the DUE assignment. In the static network equilibrium problems, provided both the cost function $\mathbf{c}$ and the demand function $\mathbf{q}$ are invertible (i.e. the functions $\mathbf{y} = \mathbf{c}^{-1}(\mathbf{c})$ and $\boldsymbol{\tau} = \mathbf{q}^{-1}(\mathbf{q})$ exist), we can obtain the following three types of formulations:

(Primal VIP)

Find $(\mathbf{y}^*, \mathbf{q}^*) \in K_P$ such that $\mathbf{c}(\mathbf{y}^*)\cdot(\mathbf{y}-\mathbf{y}^*) + \mathbf{q}^{-1}(\mathbf{q}^*)\cdot(\mathbf{q}-\mathbf{q}^*) \geq \mathbf{0} \quad \forall(\mathbf{y},\mathbf{q}) \in K_P$ (4.25a)

where $K_P = \{(\mathbf{y},\mathbf{q}) \mid \mathbf{q} = \mathrm{A}\mathbf{y},\ \mathbf{y} \geq \mathbf{0},\ \mathbf{q} \geq \mathbf{0}\}$

(Dual VIP)

Find $(\mathbf{c}^*, \boldsymbol{\tau}^*) \in K_D$ such that $\mathbf{c}^{-1}(\mathbf{c}^*)\cdot(\mathbf{c}-\mathbf{c}^*) + \mathbf{q}(\boldsymbol{\tau}^*)\cdot(\boldsymbol{\tau}-\boldsymbol{\tau}^*) \geq \mathbf{0} \quad \forall(\mathbf{c},\boldsymbol{\tau}) \in K_D$ (4.25b)

where $K_D = \{(\mathbf{c},\boldsymbol{\tau}) \mid \mathbf{c} = \mathrm{A}^T\boldsymbol{\tau},\ \mathbf{c} \geq \mathbf{c}_{\min},\ \boldsymbol{\tau} \geq \mathbf{0}\}$

(Primal-Dual VIP)

Find $(\mathbf{y}^*, \boldsymbol{\tau}^*) \in K_{PD}$ such that

$$(\mathbf{c}(\mathbf{y}^*) + \mathrm{A}^T\boldsymbol{\tau}^*)\cdot(\mathbf{y}-\mathbf{y}^*) + (\mathbf{q}(\boldsymbol{\tau}^*) - \mathrm{A}\mathbf{y}^*)\cdot(\boldsymbol{\tau}-\boldsymbol{\tau}^*) \geq \mathbf{0} \quad \forall(\mathbf{y},\boldsymbol{\tau}) \in K_{PD} \tag{4.25c}$$

where $K_{PD} = \{(\mathbf{y},\boldsymbol{\tau}) \mid \mathbf{y} \geq \mathbf{0},\ \boldsymbol{\tau} \geq \mathbf{0}\}$

On the other hands, the DUE assignment based on link-node variables can be converted into only "Primal-Dual" type even if $\mathbf{c}$ and $\mathbf{q}$ are invertible; neither "Primal" nor "Dual" type are possible. The reason is that the link cost function in the DUE assignment depends not only the flow vector $\mathbf{y}$ but also the node-arrival time $\boldsymbol{\tau}$, which destroys a certain "bisymmetry" structure of the problem.

4.2 Nonlinear Complementarity and Fixed Point Formulations

For the convenience of the standard NCP formulation presented below, we introduce a new variable, $\hat{\tau}_i^u \equiv \hat{\tau}_i(u) \equiv u - \tau_i(u)$, which means the equilibrium travel time from node *i* to a destination for each destination arrival time *u*. Then, the link cost can be represented as a function of $\hat{\tau}_i^u$:

$$c_{ij}^u(y_{ij}^u, \tau_i^u) = Max[\alpha_{ij} y_{ij}^u + \beta_{ij}^u - (u - \hat{\tau}_i^u),\ m_{ij}] \equiv \hat{c}_{ij}^u(y_{ij}^u, \hat{\tau}_i^u)\,. \tag{4.26}$$

We also let $\hat{q}_{od}^u(\hat{\boldsymbol{\tau}}_o)$ denote the OD demand as a function of $\hat{\tau}_i^u$.

1) Deterministic Demand Function Case

Using the new variables defined above, the complementarity conditions for the DUE assignment presented in the section 3, (3.5) and (3.7), are transformed into:

$$\begin{cases} y_{ij}^u \cdot (\hat{c}_{ij}^u - \hat{\tau}_i^u + \hat{\tau}_j^u) = 0 \\ \quad \hat{c}_{ij}^u - \hat{\tau}_i^u + \hat{\tau}_j^u \ \geq 0, \quad y_{ij}^u \geq 0 \end{cases} \qquad \forall (i,j) \in L\,, \quad \forall u \in \mathcal{T} \tag{3.5'}$$

$$\begin{cases} q_{od}^u \cdot \{\hat{\tau}_o^u + \psi_{od}^u - \rho_{od}\} = 0 \\ \quad \hat{\tau}_o^u + \psi_{od}^u - \rho_{od} \ \geq 0, \quad q_{od}^u \geq 0 \end{cases} \qquad \forall (o,d) \in \mathcal{P},\ \forall u \in \mathcal{T} \tag{3.7'}$$

Next, consider the following complementarity conditions instead of the flow conservation equations (3.2b) and (3.8) :

$$\begin{cases} \hat{\tau}_k^u \cdot (\sum_j y_{kj}^u - \sum_i y_{ik}^u - q_{kd}^u) = 0 \\ \quad \sum_j y_{kj}^u - \sum_i y_{ik}^u - q_{kd}^u \ \geq 0, \quad \hat{\tau}_k^u \geq 0 \end{cases} \qquad \forall k \in N\,, \qquad \forall u \in \mathcal{T} \tag{3.2'}$$

$$\begin{cases} \rho_{od} \cdot (\sum_{u \in \mathcal{T}} q_{od}^u - Q_{od}) = 0 \\ \quad \sum_{u \in \mathcal{T}} q_{od}^u - Q_{od} \ \geq 0, \quad \rho_{od} \geq 0 \end{cases} \qquad \forall (o,d) \in \mathcal{P},\ \forall u \in \mathcal{T} \tag{3.8'}$$

This leads us to the subsequent theorem.

Theorem 4.3. *Suppose that* $q_{od} \geq 0$, $\psi_{od}^u > 0$ $\forall (o,d) \in \mathcal{P}, \forall u \in \mathcal{T}$ *and* $\hat{c}_{ij} > 0$ $\forall (i,j) \in L$. *Then, the vector* $\mathbf{X}^* \in K_{1+} = \boldsymbol{R}_+^{L\times K} \times \boldsymbol{R}_+^{N\times K} \times \boldsymbol{R}_+^{M\times K} \times \boldsymbol{R}_+^{M}$ *is a solution of simultaneous DUE assignment with deterministic OD demand if and only if it satisfies the following standard NCP:*

$$\textit{Find a vector } \mathbf{X}^* \in K_{1+} \textit{ such that } \quad \mathbf{X}^* \cdot \mathbf{F}(\mathbf{X}^*) = \mathbf{0},\ \mathbf{X}^* \geq \mathbf{0},\ \mathbf{F}(\mathbf{X}^*) \geq \mathbf{0}\,, \tag{4.27a}$$

where $\mathbf{X} \in K_{1+}$ *and* $\mathbf{F}(\mathbf{X}): K_{1+} \to K_{1+}$ *are defined by*

$$\mathbf{X} = \begin{bmatrix} \mathbf{y} \\ \hat{\boldsymbol{\tau}} \\ \mathbf{q} \\ \boldsymbol{\rho} \end{bmatrix}, \quad \mathbf{F}(\mathbf{X}) = \begin{bmatrix} 0 & -\mathrm{A_K}^T & 0 & 0 \\ \mathrm{A_K} & 0 & -\mathrm{I} & 0 \\ 0 & \mathrm{I} & 0 & -\mathrm{D}^T \\ 0 & 0 & \mathrm{D} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \hat{\boldsymbol{\tau}} \\ \mathbf{q} \\ \boldsymbol{\rho} \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{c}}(\mathbf{y}, \hat{\boldsymbol{\tau}}) \\ \mathbf{0} \\ \boldsymbol{\psi} \\ -\mathbf{Q} \end{bmatrix} \tag{4.27b}$$

Proof: Since it is self-evident that any solution of the DUE assignment satisfies the NCP above, we will show that any solution to the NCP is a solution to the DUE assignment.

First, suppose that there is an $(\mathbf{y},\hat{\boldsymbol{\tau}})$ satisfying (4.27a), but that

$$\sum_j y_{kj}^u - \sum_i y_{ik}^u - \hat{q}_{kd}^u > 0 \quad \text{for some } \{ k \in N, k \neq d,\ \ u \in \mathscr{T}\}. \tag{4.28}$$

Then $\hat{\tau}_k^u \cdot (\sum_j y_{kj}^u - \sum_i y_{ik}^u - \hat{q}_{kd}^u) = 0$ implies that $\hat{\tau}_k^u = 0$. The (4.28) also implies that $y_{kj}^u > 0$ for some $\{(k,j), u\}$ since $q_{kd}^u \geq 0$, $\sum_i y_{ik}^u \geq 0$ and $y_{kj}^u \geq 0$. On the other hands, for this $\{(k,j), u\}$ the equation $y_{kj}^u \cdot (\hat{c}_{kj}^u - \hat{\tau}_k^u + \hat{\tau}_j^u) = 0$ implies that $\hat{c}_{kj}^u - \hat{\tau}_k^u + \hat{\tau}_j^u = 0$. But since $\hat{\tau}_k^u = 0$, $\hat{c}_{kj}^u = -\hat{\tau}_j^u \leq 0$, which contradicts the assumption $\hat{c}_{kj}^u > 0$.

Next, suppose that there is an $(\mathbf{q},\rho)$ satisfying (4.27a), but that

$$\sum_{u \in \mathscr{T}} q_{od}^u - Q_{od} > 0 \quad \text{for some } \{ (o,d) \in \mathscr{P},\ \ u \in \mathscr{T}\}. \tag{4.29}$$

Then $\rho_{od} \cdot (\sum_{u \in \mathscr{T}} q_{od}^u - Q_{od}) = 0$ implies that $\rho_{od} = 0$. The (4.29) also implies that $q_{od}^u > 0$ for some $\{(o,d), u\}$ since $q_{kd}^u \geq 0$ and $Q_{od} > 0$. On the other hands, for this $\{(o,d), u\}$ the equation $q_{od}^u \cdot \{\hat{\tau}_o^u + \psi_{od}^u - \rho_{od}\} = 0$ implies that $\hat{\tau}_o^u + \psi_{od}^u - \rho_{od} = 0$. But since $\rho_{od} = 0$, $\psi_{od}^u = -\hat{\tau}_o^u \leq 0$, which contradicts the assumption $\psi_{od}^u > 0$. This completes the proof.□

Interestingly, the NCP (4.27) further reduces to the *Linear* Complementarity Problem (LCP). Note here that the link cost function (4.26) is equivalent to the following complementarity condtions:

$$\begin{cases} (c_{ij}^u - m_{ij}) \cdot \{c_{ij}^u - (\alpha_{ij} y_{ij}^u + \beta_{ij}^u + \hat{\tau}_i^u)\} = 0 \\ \qquad c_{ij}^u - (\alpha_{ij} y_{ij}^u + \beta_{ij}^u + \hat{\tau}_i^u) \geq 0, \quad c_{ij}^u \geq m_{ij} \end{cases} \quad \forall (i,j) \in L,\ \ \forall u \in \mathscr{T} \tag{4.30}$$

where the c_{ij}^u's are regarded as unknown *variables* in the system of equations and inequalities. Introducing new variables $e_{ij}^u \equiv c_{ij}^u - m_{ij}\ \forall (i,j) \in L, \forall u \in \mathscr{T}$, the (4.30) can be transformed into

$$\begin{cases} e_{ij}^u \cdot \{e_{ij}^u - (\alpha_{ij} y_{ij}^u + \hat{\tau}_i^u + e_{ij}^{u-1} - \hat{\tau}_i^{u-1} - du)\} = 0 \\ \qquad e_{ij}^u - (\alpha_{ij} y_{ij}^u + \hat{\tau}_i^u + e_{ij}^{u-1} - \hat{\tau}_i^{u-1} - du \geq 0, \quad e_{ij}^u \geq 0 \end{cases} \quad \forall (i,j) \in L,\ \ \forall u \in \mathscr{T} \tag{4.31a}$$

This can be represented as the following vector-matrix form:

$$\begin{cases} \mathbf{e}^u \bullet \{\mathbf{M}(\mathbf{e}^u - \mathbf{e}^{u-1}) - \mathbf{y}^u - \mathbf{M}\mathbf{A}_+^T(\hat{\boldsymbol{\tau}}^u - \hat{\boldsymbol{\tau}}^{u-1}) - \boldsymbol{\mu}\} = \mathbf{0} \\ \qquad \mathbf{M}(\mathbf{e}^u - \mathbf{e}^{u-1}) - \mathbf{y}^u - \mathbf{M}\mathbf{A}_+^T(\hat{\boldsymbol{\tau}}^u - \hat{\boldsymbol{\tau}}^{u-1}) - \boldsymbol{\mu} \geq \mathbf{0}, \quad \mathbf{e}^u \geq \mathbf{0} \end{cases} \quad \forall u \in \mathscr{T} \tag{4.31b}$$

or equivalently,

$$\begin{cases} \mathbf{e} \bullet (\mathbf{M}_K \mathbf{e} - \mathbf{y} - \mathbf{M}_K \mathbf{A}_{K+}^T \hat{\boldsymbol{\tau}} - \boldsymbol{\mu}_K) = \mathbf{0} \\ \qquad \mathbf{M}_K \mathbf{e} - \mathbf{y} - \mathbf{M}_K \mathbf{A}_{K+}^T \hat{\boldsymbol{\tau}} - \boldsymbol{\mu}_K \geq \mathbf{0}, \quad \mathbf{e} \geq \mathbf{0} \end{cases} \tag{4.31c}$$

where the following notation is used: $\mathbf{e}^u = (.., e_{ij}^u ,..)^T \in \boldsymbol{R}^{\mathrm{L}}$; $\mathbf{e} = (\mathbf{e}^0 ..., \mathbf{e}^u, ... \mathbf{e}^{\mathrm{K}})^T \in \boldsymbol{R}^{\mathrm{L \cdot K}}$; $\boldsymbol{\mu} = (.., \mu_{ij}^* ,..)^T \in \boldsymbol{R}^{\mathrm{L}}$; $\boldsymbol{\mu}_{\mathrm{K}} = (\boldsymbol{\mu} ..., \boldsymbol{\mu}, ... \boldsymbol{\mu})^T \in \boldsymbol{R}^{\mathrm{L \cdot K}}$; M= $\mathrm{L} \times \mathrm{L}$ diagonal matrix with entries μ_{ij}^* / du; $\mathrm{M_K}$ = a block diagonal matrix with K diagonal blocks all equal to [-M, M]; A_+ = a matrix that consists of -1 entries of link-node incidence matrix A; $\mathrm{A}_{\mathrm{K+}}$= a block diagonal matrix with K diagonal blocks all equal to A_+.

Thus, we have the following LCP representation for the DUE assignment:

Theorem 4.4. *Suppose that* $q_{od} \geq 0 \quad \forall (o,d) \in \mathscr{P}$*. Then, the vector* $\mathbf{X}^* \in K_{1\mathrm{L}+} = \boldsymbol{R}_+^{\mathrm{L}\times\mathrm{K}} \times \boldsymbol{R}_+^{\mathrm{L}\times\mathrm{K}} \times \boldsymbol{R}_+^{\mathrm{N}\times\mathrm{K}} \times \boldsymbol{R}_+^{\mathrm{M}\times\mathrm{K}} \times \boldsymbol{R}_+^{\mathrm{M}}$ *is a solution of simultaneous DUE assignment with deterministic OD demand if and only if it satisfies the following standard LCP:*

$$\textit{Find a vector } \mathbf{X}^* \in K_{1\mathrm{L}+} \textit{ such that } \mathbf{X}^* \bullet \mathbf{F}_{\mathrm{L}}(\mathbf{X}^*) = \mathbf{0},\ \mathbf{X}^* \geq \mathbf{0},\ \mathbf{F}_{\mathrm{L}}(\mathbf{X}^*) \geq \mathbf{0}, \tag{4.31a}$$

where $\mathbf{X} \in K_{1\mathrm{L}+}$ *and* $\mathbf{F}_{\mathrm{L}}(\mathbf{X}): K_{1\mathrm{L}+} \rightarrow K_{1\mathrm{L}+}$ *are defined by*

$$\mathbf{X} = \begin{bmatrix} \mathbf{e} \\ \mathbf{y} \\ \hat{\tau} \\ \mathbf{q} \\ \rho \end{bmatrix}, \quad \mathbf{F}_{\mathrm{L}}(\mathbf{X}) = \begin{bmatrix} \mathrm{M_K} & -\mathrm{I} & \mathrm{M_K A_{K+}}^T & 0 & 0 \\ \mathrm{I} & 0 & -\mathrm{A_K}^T & 0 & 0 \\ 0 & \mathrm{A_K} & 0 & -\mathrm{I} & 0 \\ 0 & 0 & \mathrm{I} & 0 & -\mathrm{D}^T \\ 0 & 0 & 0 & \mathrm{D} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{y} \\ \hat{\tau} \\ \mathbf{q} \\ \rho \end{bmatrix} + \begin{bmatrix} -\mu \\ \mathbf{m} \\ \mathbf{0} \\ \psi \\ -\mathbf{Q} \end{bmatrix} \tag{4.31b}$$

2) Stochastic Demand Function Case

For the LOGIT type stochastic demand model, we can not derive a standard NCP representation as in the previous (deterministic) case. The reason is that the $\rho_{od}^{u,v}$ can be negative and the flow conservation equation (4.20b) can not be replaced with the complementarity condition as in (3.8'). As for the general demand model presented in Theorem 4.2B, however, we can constitute a standard NCP.

Theorem 4.5. *Suppose that* $\hat{q}_{od}^u \geq 0 \ \forall (o,d) \in \mathscr{P}, \forall u \in \mathscr{T}$ *and* $\hat{c}_{ij}^u > 0 \ \forall (i,j) \in L, \forall u \in \mathscr{T}$*. Then, the vector* $\mathbf{x}^* \in K_{2B+} = \boldsymbol{R}_+^{\mathrm{L}\times\mathrm{K}} \times \boldsymbol{R}_+^{\mathrm{N}\times\mathrm{K}}$ *is a solution of simultaneous DUE assignment with general OD demand for a time period* $\mathscr{T}$ *if and only if it satisfies the following standard NCP:*

$$\textit{Find a vector } \mathbf{x}^* \in K_{2B+} \textit{ such that } \mathbf{x}^* \cdot \mathbf{F}(\mathbf{x}^*) = \mathbf{0},\ \mathbf{x}^* \geq \mathbf{0},\ \mathbf{F}(\mathbf{x}^*) \geq \mathbf{0}. \tag{4.32}$$

where $\mathbf{x} \in K_{2B+}$ and $\mathbf{F}(\mathbf{x})$: $K_{2B+} \rightarrow K_{2B+}$ are defined as

$$\mathbf{x} \equiv \begin{bmatrix} \mathbf{y} \\ \hat{\tau} \end{bmatrix}, \quad \mathbf{F}(\mathbf{x}) \equiv \begin{bmatrix} \hat{\mathbf{c}}(\mathbf{y}, \hat{\tau}) \\ -\hat{\mathbf{q}}(\hat{\tau}) \end{bmatrix} + \begin{bmatrix} 0 & -\mathrm{A_K}^T \\ \mathrm{A_K} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \hat{\tau} \end{bmatrix}. \tag{4.33}$$

As is well known in the mathematical programming theory, the NCP formulation above implies that the DUE assignment can also be represented as a simple fixed point problem. To show this, define a mapping $\mathbf{H}(\mathbf{x})$: $K_{2B+} \rightarrow K_{2B+}$ as

$$\mathbf{H}(\mathbf{x}) = (\ldots, H_i, \ldots) = [\mathbf{x} - \mathrm{G}^{-1}\mathbf{F}(\mathbf{x})]_+, \tag{4.34}$$

where for any vector $\mathbf{z} \in R^n$, $[\mathbf{z}]_+$ denotes the vector with components $\max[0, z_i]$, i=1,2,...n, G is an $(\mathrm{KL}+\mathrm{KN}) \times (\mathrm{KL}+\mathrm{KN})$ diagonal matrix with positive entries. Then the following theorem holds.

Theorem 4.6. *Suppose that* $\hat{q}_{od}^u \geq 0 \ \forall (o,d) \in \mathscr{P}, \forall u \in \mathscr{T}$ *and* $\hat{c}_{ij}^u > 0 \ \forall (i,j) \in L, \forall u \in \mathscr{T}$*. Then, the vector* $\mathbf{x}^* \in K_{2B+}$ *is a solution of simultaneous DUE assignment with general OD demand for a time period* $\mathscr{T}$ *if and only if it satisfies the following fixed point problem:*

$$\textit{Find a vector } \mathbf{x}^* \in K_{2B+} \textit{ such that } \mathbf{x}^* = \mathbf{H}(\mathbf{x}^*) \tag{4.35}$$

The proof is elementary and it is omitted here.

5. Existence and Uniqueness Analyses

In this section we analyze the existence and uniqueness properties of DUE assignment based on the variational inequality formulation obtained in the previous section.

5.1 Existence

In the previous studies, we proved the existence of the DUE assignment when OD demands are fixed (Kuwahara and Akamatsu(1993), Akamatsu and Kuwahara(1994)). The proof was based on the Brouwer's / Kakutani's existence theorem for a fixed point problem. Note that the Brouwer's / Kakutani's theorem requires the compactness of the feasible set, which was satisfied in the fixed demand case. This is not, however, satisfied when the OD flows are functions of $\boldsymbol{\tau}_o$, since the feasible set is not necessarily bounded. Therefore, it is difficult to extend our previous approach to the current elastic demand case in a straight-forward manner.

Thus, we show the existence by the VIP formulation, where the following lemma is useful:

Lemma 5.1. *(Kinderlehrer and Stampacchia(1980)) Let* $K \subset R^n$ *be closed and convex and* $\mathbf{F}: K \rightarrow K$ *be continuous. We set* $K_r = K \cap B_r(0)$ *where* $B_r(0)$ *is the closed ball of radius r and center* $0 \in R^n$. *There exist a solution to* $VI(K, \mathbf{F})$ *if and only if there exists an* $r > 0$ *such that a solution* $\mathbf{x}_r^* \in K_r$ *of* $VI(K_r, \mathbf{F})$ *satisfies* $\|\mathbf{x}_r^*\| < r$.

Defining the set $\overline{B}_r = \{(\mathbf{y}, \boldsymbol{\tau}) \mid 0 \leq y_{ij} \leq r_y \ \ \forall (i,j) \in L,\ \ u - r_\tau \leq \tau_k \leq r_\tau \ \ \forall k \in N\}$ and $K_{S\,r} \equiv K_S \cap \overline{B}_r$, the above lemma immediately yields the following result:

Lemma 5.2. *There exist a solution to the DUE assignment with separable demand functions if and only if there exist* $r_y > 0$ *and* $r_\tau > 0$ *such that a solution* $\mathbf{x}_r^* \in K_{S\,r}$ *of* $VI(K_{S\,r}, \mathbf{F})$ *satisfies* $0 \leq y_{ij} \leq r_y \ \ \forall ij \in L,\ \ u - r_\tau \leq \tau_k \leq u \ \ \forall k \in N$.

The lemma enables us to establish the more convenient existence theorem:

Theorem 5.1. *Suppose that there exist positive constants r_1 and r_2, such that*

$$c_{ij}(y_{ij}, \tau_i) \geq r_1 \quad \forall (i,j) \in L,\ \ \forall (y_{ij}, \tau_i) \in K_S, \tag{5.1}$$

and

$$q_{od}(\tau_o) < r_2 \quad \forall od\ ,\ \forall \tau_o \geq r_2. \tag{5.2}$$

Then, the DUE assignment with separable demand functions has at least one solution.

Since the proof almost parallels that of the static equilibrium assignment represented as a VIP (see, for example, Theorem 4.3 in Nagurney(1993)) and somewhat lengthy, we omit here.

For the non-separable demands case, the similar argument holds.

Theorem 5.2. *Suppose that there exist positive constants r_1 and r_2, such that*

$$c_{ij}^u(y_{ij}^u, \tau_i^u) \geq r_1 \quad \forall (i,j) \in L, \forall u \in U,\ \ \forall (y_{ij}^u, \tau_i^u) \in K_S, \tag{5.3}$$

and

$$q_{od}^u(\ldots, \tau_o^u, \ldots) < r_2 \quad \forall od\ ,\ \ \forall u \in U,\ \forall \tau_o^u \geq r_2. \tag{5.4}$$

Then, the DUE assignment with non- separable demand functions has at least one solution.

5.2 Uniqueness

We shall examine the uniqueness property of the DUE assignment relying on the following lemmas that are basic in the variational inequality theory:

Lemma 5.3. *Suppose that* $\mathbf{F}(\mathbf{x})$ *is continuously differentiable on K and the Jacobian matrix is positive definite. Then* $\mathbf{F}(\mathbf{x})$ *is strictly monotone.*

Lemma 5.4. *Suppose that* $\mathbf{F}(\mathbf{x})$ *is strictly monotone on K. Then the solution of* $VI(K, \mathbf{F})$ *is unique, if one exists.*

1) Deterministic Demand Function Case

From the equivalency between VIP and NCP, the lemma above can also be applied to the NCP/LCP representation of the DUE assignment. Therefore, we analyze the uniqueness by using the mapping $\mathbf{F}_{\mathrm{L}}(\mathbf{X})$ in LCP formulation presented in section 4.2. The Jacobian of the mapping $\mathbf{F}_{\mathrm{L}}(\mathbf{X})$ defined in (4.31b) yields

$$\nabla \mathbf{F}_{\mathrm{L}}(\mathbf{X}) = \begin{bmatrix} 0 & -\mathrm{I} & 0 & 0 & 0 \\ \mathrm{I} & 0 & -\mathrm{A}_{\mathrm{K}}{}^{T} & 0 & 0 \\ 0 & \mathrm{A}_{\mathrm{K}} & 0 & -\mathrm{I} & 0 \\ 0 & 0 & \mathrm{I} & 0 & -\mathrm{D}^{T} \\ 0 & 0 & 0 & \mathrm{D} & 0 \end{bmatrix} + \begin{bmatrix} \mathrm{M}_{\mathrm{K}} & 0 & \mathrm{M}_{\mathrm{K}}\mathrm{A}_{\mathrm{K+}}{}^{T} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.5}$$

The first matrix of r.h.s. in (5.5) is a bisymmetric matrix, whose components mutually cancel out in a quadratic form of the matrix. Therefore, for a vector $\mathbf{x} = [\mathbf{C}, \mathbf{Y}, \mathbf{T}, \mathbf{Q}, \mathbf{P}]^{T} \in K = \boldsymbol{R}^{2\mathrm{L}\times\mathrm{K}+\mathrm{N}\times\mathrm{K}+\mathrm{M}\times\mathrm{K}+\mathrm{M}}$,

$$\begin{aligned} \mathbf{x}^{T}\nabla \mathbf{F}_{\mathrm{L}}(\mathbf{X})\,\mathbf{x} &= \sum_{u}\sum_{ij} C_{ij}^{u}\cdot(\mu_{ij}^{*}/du)\cdot\{(C_{ij}^{u}-C_{ij}^{u-1})-(T_{i}^{u}-T_{i}^{u-1})\} \\ &= \sum_{ij}(\mu_{ij}^{*}/du)\sum_{u}(C_{ij}^{u})^{2} + \sum_{ij}(\mu_{ij}^{*}/du)\sum_{u} C_{ij}^{u}\cdot(T_{i}^{u}-T_{i}^{u-1}-C_{ij}^{u-1}) \end{aligned} \tag{5.6}$$

Although the first term of r.h.s. in (5.6) is positive for any vector $\mathbf{x} = [\mathbf{C}, \mathbf{Y}, \mathbf{T}, \mathbf{Q}, \mathbf{P}]^{T} \in K$ ($\because$ $\mu_{ij}^{*} > 0$ $\forall (i, j) \in L$), $\mathbf{x}^{T}\nabla \mathbf{F}_{\mathrm{L}}(\mathbf{X})\,\mathbf{x}$ can be either positive or negative due to the existence of the second term. That is, we can not be assure the positive definiteness of the Jacobian $\nabla \mathbf{F}_{\mathrm{L}}(\mathbf{X})$.

2) Stochastic Demand Function Case

When we replace the link cost function in the VIP (4.22) (i.e. LOGIT type demand function case) with the linear complementarity conditions (4.31) , the mapping $\mathbf{F}(\mathbf{X})$ in (4.21) reduces to

$$\mathbf{X} = \begin{bmatrix} \mathbf{e} \\ \mathbf{y} \\ \hat{\boldsymbol{\tau}} \\ \mathbf{q} \\ \boldsymbol{\rho} \end{bmatrix}, \quad \mathbf{F}(\mathbf{X}) = \begin{bmatrix} \mathrm{M}_{\mathrm{K}} & -\mathrm{I} & \mathrm{M}_{\mathrm{K}}\mathrm{A}_{\mathrm{K+}}{}^{T} & 0 & 0 \\ \mathrm{I} & 0 & -\mathrm{A}_{\mathrm{K}}{}^{T} & 0 & 0 \\ 0 & \mathrm{A}_{\mathrm{K}} & 0 & -\mathrm{E}_{\mathrm{K}}^{T} & 0 \\ 0 & 0 & \mathrm{E}_{\mathrm{K}}^{T} & 0 & -\mathrm{D}^{T} \\ 0 & 0 & 0 & \mathrm{D} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{y} \\ \hat{\boldsymbol{\tau}} \\ \mathbf{q} \\ \boldsymbol{\rho} \end{bmatrix} + \begin{bmatrix} -\boldsymbol{\mu} \\ \mathbf{m} \\ \mathbf{0} \\ \mathrm{E}_{\mathrm{K}}^{T}\mathbf{u} - \boldsymbol{\psi} + (1/\theta)\ln\mathbf{q} \\ -\mathbf{w} \end{bmatrix}. \tag{5.7}$$

Similar to the deterministic demand case, for a vector $\mathbf{x} = [\mathbf{C}, \mathbf{Y}, \mathbf{T}, \mathbf{Q}, \mathbf{P}]^{T} \in K = \boldsymbol{R}^{2\mathrm{L}\times\mathrm{K}+\mathrm{N}\times\mathrm{K}+\mathrm{M}\times\mathrm{K}\times\mathrm{J}+\mathrm{M}\times\mathrm{J}}$, it follows that

$$\mathbf{x}^T \nabla \mathbf{F}(\mathbf{X})\ \mathbf{x} = \sum_{ij} \frac{\mu_{ij}^*}{du} \sum_u (C_{ij}^u)^2 + \sum_u \sum_v \sum_{od} \frac{(Q_{od}^{u,v})^2}{\theta q_{od}^{u,v}} + \sum_{ij} \frac{\mu_{ij}^*}{du} \sum_u C_{ij}^u \cdot (T_i^u - T_i^{u-1} - C_{ij}^{u-1})\,. \qquad (5.8)$$

Although the first and second terms of r.h.s. in (5.8) is positive for any vector $\mathbf{x} = [\mathbf{C}, \mathbf{Y}, \mathbf{T}, \mathbf{Q}, \mathbf{P}]^T \in K$, $\mathbf{x}^T \nabla \mathbf{F}(\mathbf{X})\ \mathbf{x}$ can be either positive or negative due to the existence of the third term. That is, we can not be assure the positive definiteness of the Jacobian $\nabla \mathbf{F}(\mathbf{X})$.

We next analyze the DUE assignment with general demand model presented in *Theorem 4.2B*. Similar to the case above, we slightly modify the original mapping $\mathbf{F}(\mathbf{x})$ (in *Theorem 4.2B*):

$$\mathbf{X} = \begin{bmatrix} \mathbf{e} \\ \mathbf{y} \\ \hat{\boldsymbol{\tau}} \end{bmatrix}, \quad \mathbf{F}(\mathbf{X}) = \begin{bmatrix} \mathrm{M_K} & -\mathrm{I} & \mathrm{M_K A_{K+}}^T \\ \mathrm{I} & 0 & -\mathrm{A_K}^T \\ 0 & \mathrm{A_K} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{y} \\ \hat{\boldsymbol{\tau}} \end{bmatrix} + \begin{bmatrix} -\boldsymbol{\mu} \\ \mathbf{m} \\ \mathbf{q}(\hat{\boldsymbol{\tau}}) \end{bmatrix} \qquad (5.9)$$

For a vector $\mathbf{x} = [\mathbf{C}, \mathbf{Y}, \mathbf{T}]^T \in K = \boldsymbol{R}^{2\mathrm{L}\times\mathrm{K}+\mathrm{N}\times\mathrm{K}}$,

$$\mathbf{x}^T \nabla \mathbf{F}(\mathbf{X})\ \mathbf{x} = \sum_{ij} \frac{\mu_{ij}^*}{du} \sum_u (C_{ij}^u)^2 + \sum_u \sum_o (T_i^u)^2 \sum_{u'} \frac{\partial q_{od}^u}{\partial \hat{\tau}_o^{u'}} - \sum_{ij} \frac{\mu_{ij}^*}{du} \sum_u C_{ij}^u \cdot (T_i^u - T_i^{u-1} - C_{ij}^{u-1}) \qquad (5.10)$$

Similar to the LOGIT type demand, $\mathbf{x}^T \nabla \mathbf{F}(\mathbf{X})\ \mathbf{x}$ in (5.10) can be either positive or negative due to the existence of the third term. Thus, we can conclude that the uniqueness of the DUE assignment is not guaranteed in general.

6. Algorithms

This section discusses the algorithms for solving the DUE assignment. Before describing the algorithm, we briefly explain the merit function (gap function) for VIP / NCP, which is a useful tool to develop globally convergent / efficient algorithms.

6.1 Equivalent Differential Optimization Problem

A merit function for a variational inequality problem $VI(K, \mathbf{F})$ is a non-negative function $f(\mathbf{x})$ such that $\mathbf{x}^*$ is a solution of the $VI(K, \mathbf{F})$ if and only if $f(\mathbf{x}^*) = 0$ and $\mathbf{x}^* \in K$, i.e. the global solution of the problem $MP(K, f): \min. f(\mathbf{x})$ subject to $\mathbf{x} \in K$, is a solution of $VI(K, \mathbf{F})$. A merit function for a standard NCP also can be defined in the similar manner.

Here we introduce two kinds of merit functions: one is the regularized gap function defined for variational inequalities, which is proposed by Fukushima. The other is a Fisher's merit function for nonlinear complementarities.

1) Fukushima's Merit Function for VIP and NCP

Let K be a closed and convex subset of R^n and let $\mathbf{F}$ be a mapping from R^n into itself. Fukushima (1992) showed that $VI(K, \mathbf{F}(\mathbf{x}))$ is equivalent to the following optimization problem with a differentiable objective function:

$$\min.\ f_0(\mathbf{x}) \equiv -\mathbf{F}(\mathbf{x})\cdot(\mathbf{z}-\mathbf{x}) - \frac{1}{2}(\mathbf{z}-\mathbf{x})\cdot \mathrm{G}(\mathbf{z}-\mathbf{x}) \quad \text{subject to } \mathbf{x}\in K, \tag{6.1}$$

where G is an $n \times n$ symmetric positive definite matrix and $\mathbf{z}$ is a solution of the following problem:

$$\min. \mathbf{F}(\mathbf{x})\cdot(\mathbf{z}-\mathbf{x}) + \frac{1}{2}(\mathbf{z}-\mathbf{x})\cdot \mathrm{G}(\mathbf{z}-\mathbf{x}) \quad \text{subject to } \mathbf{z}\in K. \tag{6.2}$$

The optimization problem (6.2) means that $\mathbf{z}$ is a projection of the point $\mathbf{x} - \mathrm{G}^{-1}\mathbf{F}(\mathbf{x})$ onto K with respect to the norm $\|\cdot\|_G$, where $\|\mathbf{x}\|_G^2 \equiv \mathbf{x}\cdot \mathrm{G}\mathbf{x}$. We denote it as $\mathbf{z} = \mathrm{Proj}_{K,G}(\mathbf{x} - \mathrm{G}^{-1}\mathbf{F}(\mathbf{x}))$.

In the special case where $K = R^n_+$, nonnegative orthant in R^n, $VI(K, \mathbf{F}(\mathbf{x}))$ can be rewritten as the standard NCP and the projection (6.2) simply reduces to

$$\mathbf{z} = \mathrm{Proj}_{K,G}(\mathbf{x} - \mathrm{G}^{-1}\mathbf{F}(\mathbf{x})) = [\mathbf{x} - \mathrm{G}^{-1}\ \mathbf{F}(\mathbf{x})]_+, \tag{6.3}$$

where for any vector $\mathbf{z}\in R^n$, $[\mathbf{z}]_+$ denotes the vector with components $\max[0, z_i]$, i=1,2,...n.

The merit function f_0 has the following nice properties: 1) if $\mathbf{F}(\mathbf{x})$ is continuously differentiable then $f_0(\mathbf{x})$ is also continuously differentiable, 2) if $\mathbf{F}(\mathbf{x})$ is strictly monotone then every stationary point of $MP(K, f)$ is a global minimum point of $MP(K, f)$.

Applying this merit function to the DUE assignment formulated as a NCP, we have the result that a vector $(\mathbf{y}^*, \hat{\boldsymbol{\tau}}^*)$ solves the DUE assignment if and only if $(\mathbf{y}^*, \hat{\boldsymbol{\tau}}^*)$ is a global minimizer of the following optimization problem:

$$\begin{aligned} \min.\ f_1(\mathbf{y}, \hat{\boldsymbol{\tau}}) &= \sum_{u\in U}\sum_{ij\in L} \frac{1}{\gamma^u_{ij}}\left(\hat{g}^{u\,2}_{ij} - (\max[0, \hat{g}^u_{ij} - \gamma_{ij} y^u_{ij}])^2\right) \\ &+ \sum_{u\in U}\sum_{k\in N} \frac{1}{\gamma^u_k}\left(\hat{h}^{u\,2}_k - (\max[0, \hat{h}^u_k - \gamma_k \hat{\tau}^u_k])^2\right) \end{aligned} \tag{6.4}$$

subject to $\mathbf{y} \geq \mathbf{0}$, $\hat{\boldsymbol{\tau}} \geq \mathbf{0}$,

where $\hat{g}^u_{ij}(\mathbf{y}, \hat{\boldsymbol{\tau}}) \equiv c^u_{ij}(y^u_{ij}, \hat{\tau}^u_i) - \hat{\tau}^u_i + \hat{\tau}^u_j$, (6.5)

$$\hat{h}^u_k(\mathbf{y}, \hat{\boldsymbol{\tau}}) \equiv \sum_j y^u_{kj} - \sum_i y^u_{ik} - \hat{q}^u_{kd}(\hat{\boldsymbol{\tau}}), \tag{6.6}$$

γ^u_{ij} and γ^u_k are given positive parameters, and $f_1(\mathbf{y}^*, \hat{\boldsymbol{\tau}}^*) = 0$. Furthermore, if $\mathbf{F} \equiv (\hat{\mathbf{g}}, \mathbf{h})$ is a strictly monotone function then every stationary point of the above optimization problem is a solution of the DUE assignment.

Note that the result above corresponds to the standard NCP formulation. When we consider the problem equivalent to the VIP formulation, the non-negativity constraints $\hat{\boldsymbol{\tau}} \geq \mathbf{0}$ drops and the objective in (6.4) is replaced with

$$f_{1-V}(\mathbf{y}, \boldsymbol{\tau}) = \sum_{u\in U}\sum_{ij\in L} \frac{1}{\gamma^u_{ij}}\left(g^{u\,2}_{ij} - (\max[0, g^u_{ij} - \gamma^u_{ij} y^u_{ij}])^2\right) + \sum_{u\in U}\sum_{k\in N} \frac{1}{\gamma^u_k} h^{u\,2}_k. \tag{6.7}$$

where $g^u_{ij}(\mathbf{y}, \boldsymbol{\tau}) \equiv c^u_{ij}(y^u_{ij}, \tau^u_i) + \tau^u_i - \tau^u_j$, (6.5')

$$h^u_k(\mathbf{y}, \boldsymbol{\tau}) \equiv \sum_i y^u_{ik} - \sum_j y^u_{kj} + q^u_{kd}(\boldsymbol{\tau}). \tag{6.6'}$$

2) Fisher's Merit Function for NCP

Consider a standard NCP(F):

$$\text{Find a vector } \mathbf{x}^* \in R^n \text{ such that } \mathbf{x}^* \cdot \mathbf{F}(\mathbf{x}^*) = \mathbf{0},\ \mathbf{x}^* \geq \mathbf{0},\ \mathbf{F}(\mathbf{x}^*) \geq \mathbf{0}. \tag{6.8}$$

For solving NCP(F), a function $\phi: R^2 \to R$ satisfying

$$\phi(x, y) = 0 \iff xy = 0,\ x \geq 0,\ y \geq 0 \tag{6.9}$$

is useful, since we can form the following system of equations equivalent to NCP(F):

$$\Phi(\mathbf{x}) \equiv (\phi(x_1, F_1), \phi(x_2, F_2), \ldots, \phi(x_n, F_n))^T = \mathbf{0}. \tag{6.10}$$

Fisher (1992) introduced the following function satisfying the property (6-9):

$$\phi(x, y) = \sqrt{x^2 + y^2} - (x + y), \tag{6.11}$$

and defined a merit function as follows:

$$\Psi(\mathbf{x}) \equiv \|\Phi(\mathbf{x})\|^2 = \sum_{i=1}^{n} \phi(x_i, F_i(\mathbf{x}))^2. \tag{6.12}$$

The merit function Ψ has the following nice properties: 1) $\Psi(\mathbf{x})$ is continuously differentiable everywhere, 2) if $\mathbf{F}(\mathbf{x})$ is a P_0-function (see Appendix) then every stationary point of $MP(K, f)$ is a global minimum point of $MP(K, f)$.

The application of the merit function $\Psi(\mathbf{x})$ to the DUE assignment immediately lead to the results that a vector $(\mathbf{y}^*, \hat{\boldsymbol{\tau}}^*)$ solves the DUE assignment if and only if $(\mathbf{y}^*, \hat{\boldsymbol{\tau}}^*)$ is a global minimizer of the following differentiable optimization problem:

$$\min.\ f_2(\mathbf{y}, \hat{\boldsymbol{\tau}}) = \sum_{u \in U} \sum_{ij \in L} \left\{ \sqrt{{y_{ij}^u}^2 + {\hat{g}_{ij}^u}^2} - \left(y_{ij}^u + \hat{g}_{ij}^u\right) + \sqrt{{\hat{\tau}_k^u}^2 + {\hat{h}_k^u}^2} - \left(\hat{\tau}_k^u + \hat{h}_k^u\right) \right\} \tag{6.13}$$

$$\text{subject to } \mathbf{y} \geq \mathbf{0},\ \ \hat{\boldsymbol{\tau}} \geq \mathbf{0},$$

where $\hat{g}_{ij}^u$ and $\hat{h}_k^u$ are defined by (6.5) and (6.6), respectively, and $f_2(\mathbf{y}^*, \hat{\boldsymbol{\tau}}^*) = 0$. Furthermore, if $\mathbf{F} \equiv (\hat{\mathbf{g}}, \hat{\mathbf{h}})$ is a P_0-function then every stationary point of the above optimization problem is a solution of the DUE assignment.

6.2 Algorithms

We suggest two algorithms for solving the DUE assignment. First algorithm is a direct application of Fukushima's method. The method is a variant of the projection method incorporating a line search step based on the merit function (6.4). Although this algorithm is very simple and easy to implement for the DUE assignment, the strict monotonicity of the mapping $\mathbf{F}(\mathbf{x})$ is required to guarantee the theoretical convergence to the solution. Since the strict monotonicity of $\mathbf{F}(\mathbf{x})$ is not necessarily guaranteed for the DUE assignment, there is a possibility that the method fails to obtain the DUE solution. Second algorithm is an application of the method proposed by Facchinei and

Soares(1995) that is a variants of Newton's method utilizing Fisher's merit function. The convergence condition of the algorithm is rather mild.

1) Projection method using Fukushima's Merit Function

First method uses the vector

$$\mathbf{d} = \mathbf{z} - \mathbf{x} = [\mathbf{x} - \mathrm{G}^{-1}\,\mathbf{F}(\mathbf{x})]_{+} - \mathbf{x} \tag{6.14}$$

as a search direction at $\mathbf{x}$. If $\mathbf{F}(\mathbf{x})$ is a strictly monotone mapping, the vector $\mathbf{d}$ is a descent direction of the objective $f_1(\mathbf{y}, \hat{\tau})$ (for the proof, see Fukushima(1992)). The algorithm generates a sequence $\{\mathbf{x}^k\}$ by the iteration

$$\mathbf{x}^{k+1} := \mathbf{x}^k + t_k \mathbf{d}^k = (1 - t_k)\mathbf{x}^k + t_k \mathbf{z}^k,\ \ k = 0,1,2,.. \tag{6.15}$$

where $t_k \in [0,1]$ are determined from a line search problem using the merit function $f_1(\mathbf{x})$:

$$\min_{t_k}.\ f_1(\mathbf{x}^k + t_k \mathbf{d}^k) \quad s.t.\ 0 \le t_k \le 1. \tag{6.16}$$

Fukushima proved that this algorithm globally converges to the unique solution when $\mathbf{F}(\mathbf{x})$ is strictly monotone. Furthermore, he proved that if $\mathbf{F}(\mathbf{x})$ is strongly monotone, some inexact line search methods such as Armijo-type step length rule also guarantee the convergence.

There are some special cases that $\mathbf{F}(\mathbf{x})$ in our DUE problem is a strictly monotone mapping as seen in section 5. Thus, we can assure that the algorithm with the exact line search converges in that cases. It is, however, unlikely that the $\mathbf{F}(\mathbf{x})$ is strongly monotone. Therefore, it seems that we had better employ the exact line search rule (6.16) in solving the DUE assignment.

2) Newton method using Fisher's Merit Function

Suppose that $\mathbf{x}^*$ is a nondegenerate solution of NCP and that we know the sets α and β of variables which are 0 or positive at $\mathbf{x}^*$, i.e. $\alpha = \{\, i \mid x_i^* = 0\}$, $\beta = \{\, i \mid x_i^* > 0\}$. We denote the corresponding partitions of $\mathbf{x}$ and $\mathbf{F}$ as $\mathbf{x} = (\mathbf{x}_\beta,\ \mathbf{x}_\alpha) = (\mathbf{x}_\beta,\ \mathbf{0})$ and $\mathbf{F} = (\mathbf{F}_\beta,\ \mathbf{F}_\alpha) = (\mathbf{0},\ \mathbf{F}_\alpha)$, respectively. Then, we obtain the solution by solving the system of equations $F_i(\mathbf{x}_\beta, \mathbf{0}) = 0$, $i \in \beta$. This system can be solved by applying Newton's method if $[\nabla_\beta \mathbf{F}_\beta(\mathbf{x}^*)]$ is not singular, where ∇_β denotes the differential operator with respect to $\mathbf{x}_\beta$. More precisely, we generate the sequence by $\mathbf{x}_\beta^{k+1} := \mathbf{x}_\beta^k + \mathbf{d}_\beta^k$, where $\mathbf{d}_\beta^k$ is the solution of the following system of linear equations:

$$[\nabla_\beta \mathbf{F}_\beta(\mathbf{x}^k)]\,\mathbf{d}_\beta^k = -\mathbf{F}_\beta(\mathbf{x}^k). \tag{6.17}$$

Although we do not know the sets α and β before obtaining the solution in general, it is expected that if we use the well approximated set in each iteration it successfully converges to the solution. Facchinei and Soares (1995) proposed to approximate the sets α and β by

$$\alpha = \{\, i \mid x_i^k \le \varepsilon F_i(x_i^k)\},\ \ \beta = \{\, i \mid x_i^k > \varepsilon F_i(x_i^k)\}, \tag{6.18}$$

where ε is a fixed positive constants. Then, the $\mathbf{d}_\alpha^k$ is determined by $\mathbf{d}_\alpha^k = -\mathbf{x}_\alpha^k$, while $\mathbf{d}_\beta^k$ is determined by the system of linear equations (6.17) where the right hand side is replaced with

$$-\mathbf{F}_\beta(\mathbf{x}^k)+[\nabla_\alpha \mathbf{F}_\beta(\mathbf{x}^k)]\,\mathbf{x}_\alpha^k .$$

To enforce global convergence of the algorithm, they developed the method combining the "local" direction search step above and a line search step based on the Fisher's merit function. The whole algorithm can be summarized as follows:

Step 0: (Initialization)

Set $k := 0$ and set the value of the parameters ($\varepsilon, \rho, p, \kappa, \sigma$) satisfying

$\varepsilon > 0,\ \rho > 0,\ p > 2,\ \kappa \in (0, 1/2),\ \sigma \in (0,1)$.

Step 1: (Stopping Test)

If the stopping criterion is satisfied, stop.

Step 2: (Direction Finding)

Calculate $\mathbf{d}_\beta^k$ by solving (6-17), and $\mathbf{d}_\alpha^k := -\mathbf{x}_\alpha^k$.

If system (6-17) is not solvable or $\mathbf{d}^k$ does not satisfy $\nabla f_2(\mathbf{x}^k)\cdot \mathbf{d}^k \le -\rho\left\|\mathbf{d}^k\right\|^p$,

set $\mathbf{d}^k := -\nabla f_2(\mathbf{x}^k)$.

Step 3: (Move with the step size of 1)

If $\mathbf{d}^k$ satisfies $f_2(\mathbf{x}^k+\mathbf{d}^k) \le \sigma f_2(\mathbf{x}^k)$, (6.19)

set $\mathbf{x}^{k+1} := \mathbf{x}^k + \mathbf{d}^k,\ k := k+1$ and go to Step 1.

Step 4: (Line Search and Move)

Find the smallest $i = 0,\ 1,\ 2, \ldots$ such that

$$f_2(\mathbf{x}^k + 2^{-i}\mathbf{d}^k) \le f_2(\mathbf{x}^k) + \kappa\, 2^{-i}\nabla f_2(\mathbf{x}^k)\cdot \mathbf{d}^k . \qquad (6.20)$$

set $\mathbf{x}^{k+1} := \mathbf{x}^k + 2^{-i}\mathbf{d}^k,\ k := k+1$ and go to Step 1.

They also proved the following results: 1) each accumulation point of the sequence $\{\mathbf{x}^k\}$ generated by the algorithm is a stationary point of Ψ ; 2) if one of the limit points of the sequence $\{\mathbf{x}^k\}$ is a b-regular solution of the NCP, then $\{\mathbf{x}^k\}\to\bar{\mathbf{x}}$; 3) every limit point $\bar{\mathbf{x}}$ is a solution of the NCP if $\Psi(\bar{\mathbf{x}})=0$; 4) every limit point $\bar{\mathbf{x}}$ is a solution of the NCP if $\mathbf{F}(\mathbf{x})$ is a P_0-function. Although we can not be sure theoretically that this algorithm also converges to a solution of the NCPs that are not P_0, their report on various numerical experiments exhibited the robustness of the algorithm; the problems that are not P_0 and the problems that are not R-regular or b-regular(for the definition, see Appendix) at the solution were successfully solved. Therefore, it is expected that the DUE assignment also can be successfully solved by this algorithm.

The efficiency of this algorithm largely depends on what degree the system of linear equations (6.17) can be solved efficiently, since the computational time expensed for the remaining steps in this algorithm is relatively small. Fortunately, the linear system (6.17) for our DUE assignment can be easily solved exploiting the network structure of the problem. For the brevity of the explanation, we show only the case that the OD demand is fixed where the assignment can be decomposed into arrival time and the link cost function is given by

$$c_{ij} = \alpha_{ij}y_{ij} + \beta_{ij} - \tau_i \qquad \forall (i,j)\in L\cap\beta .$$

In addition, we suppress the subscript β though all the link / node variables used below are restricted to the only variables included in the set β defined in (6.18). The (6.17) for the DUE assignment can be represented as

$$\begin{cases} \mathrm{D}\,\Delta\mathbf{y} + \mathrm{A}_{-}^{T}\Delta\boldsymbol{\tau} = -\bar{\mathbf{g}} \\ \mathrm{A}\,\Delta\mathbf{y} = -\bar{\mathbf{h}} \end{cases} \tag{6.22}$$

where $\mathbf{d}^k \equiv (\Delta\mathbf{y}, \Delta\boldsymbol{\tau})$, $\bar{\mathbf{g}} \equiv \mathbf{g}(\mathbf{y}^k, \boldsymbol{\tau}^k)$, $\bar{\mathbf{h}} \equiv \mathbf{h}(\mathbf{y}^k, \boldsymbol{\tau}^k)$, A = a node-link incidence matrix,
A_{-} = a end-node-link incidence matrix that consists of -1 entries of A,
D = a Jacobian of $\mathbf{c}(\mathbf{y}, \boldsymbol{\tau})$ with respect to $\mathbf{y}$, which is a diagonal matrix with entry α_{ij},

The (6.22) reduces to the linear equations with respect to only $\Delta\boldsymbol{\tau}$:

$$(\mathrm{AD}^{-1}\mathrm{A}_{-}^{T})\Delta\boldsymbol{\tau} = \mathrm{D}^{-1}\bar{\mathbf{g}} - \bar{\mathbf{h}}, \tag{6.23}$$

and then, $\Delta\mathbf{y}$ can be obtained by the following simple calculations:

$$\Delta\mathbf{y} = \mathrm{D}^{-1}(\mathrm{A}_{-}^{T}\Delta\boldsymbol{\tau} - \bar{\mathbf{g}}) \tag{6.24}$$

or $$\Delta y_{ij} = (\Delta\tau_j - \bar{g}_{ij})/\alpha_{ij} \qquad \forall (i,j) \in L \cap \beta$$

Considering the particular property of the node-link incidence matrix A, it can be shown that the matrix $\mathrm{AD}^{-1}\mathrm{A}_{-}^{T}$ is a very sparse matrix where the (*i*,*j*) entries are given by

$$\begin{cases} -1/\alpha_{ij} & \text{if there exists a link i} \to \text{j}, \\ \sum_k (1/\alpha_{ki}) - \sum_k (1/\alpha_{ik}) & \text{if i = j (diagonal entry)}, \\ 0 & \text{otherwise.} \end{cases} \tag{6.25}$$

This means that the coefficient matrix in (6.23) can be obtained with a very small computational task and it does not require large storages. Furthermore, (6.25) yields an upper triangular matrix when we have the set β consisting of only one-way links. Therefore, the system of linear equations (6.17) can be efficiently solved. Thus, we know that the algorithm for the DUE assignment is efficient and it can by applied to large scale networks.

7. Summary and Conclusions

We have considered the dynamic user equilibrium (DUE) assignment with elastic demand on an over saturated network for a many-to-one Origin-Destination pattern. First, we defined the DUE conditions for user's simultaneous choice of route and departure time, and then the basic formulation and the decomposition with respect to arrival time at a destination were shown. Next, various alternative formulations were presented: variational inequality (VI), nonlinear complementarity (NC) and fixed point (FP) formulations. Unlike the previous path-flow formulations, our formulations make use of only node / link variables, where the mapping is simple and, therefore, it is easy to examine the mathematical properties. The VI formulation enabled us to establish the existence of the equilibrium solution. Furthermore, the properties of the mapping showed that the uniqueness of the DUE assignment can not necessarily guaranteed, excluding a certain special case. Finally, some algorithms based on the VI / NC formulation of the DUE assignment were suggested. The algorithms utilize the merit functions, which is a useful tools for enforcing the global convergence or accelerating the convergence.

The following topics are left for the future researches. First, we should further investigate the conditions that are required for holding such properties as the First-In-First-Work principle peculiar to the departure time equilibrium in general networks. The results would give us the insights into the simultaneous equilibrium and would be useful for the development of the extended models / the efficient algorithms. Second, developing the robust algorithms that converge to the DUE solution under milder condition is also important. Finally, extending our framework to the many to many OD pattern is most important and challenging topic.

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Appendix

We summarize some definitions concerning the properties of a vector-valued function and NCP.

Definition 1

a) $\mathbf{F}: R^n \to R^n$ is *monotone* on a set $S \subseteq R^n$ if

$(\mathbf{x}-\mathbf{y})\cdot(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y})) \geq 0 \quad \forall \mathbf{x},\mathbf{y} \in S$.

b) $\mathbf{F}: R^n \to R^n$ is *strictly monotone* on a set $S \subseteq R^n$ if

$(\mathbf{x}-\mathbf{y})\cdot(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y})) > 0 \quad \forall \mathbf{x},\mathbf{y} \in S,\ \mathbf{x} \neq \mathbf{y}$.

c) $\mathbf{F}: R^n \to R^n$ is *strongly monotone* on a set $S \subseteq R^n$ if

$(\mathbf{x}-\mathbf{y})\cdot(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{y})) \geq \mu\|\mathbf{x}-\mathbf{y}\|^2$ for some $\mu > 0 \ \forall \mathbf{x},\mathbf{y} \in S$.

Definition 2

a) an $n \times n$ matrix M is a *semi-positive definite* matrix on a set $S \subseteq R^n$ if

$\mathbf{x}\cdot \mathrm{M}\mathbf{x} \geq 0 \quad \forall \mathbf{x} \in S$

b) an $n \times n$ matrix M is a *positive definite matrix* on a set $S \subseteq R^n$ f

$\mathbf{x}\cdot \mathrm{M}\mathbf{x} > 0 \quad \forall \mathbf{x} \in S,\ \mathbf{x} \neq \mathbf{0}$

c) an $n \times n$ matrix M is a *strongly positive definite matrix* on a set $S \subseteq R^n$ if

$\mathbf{x}\cdot \mathrm{M}\mathbf{x} \geq \mu\|\mathbf{x}\|^2$ for some $\mu > 0 \ \forall \mathbf{x} \in S,\ \mathbf{x} \neq \mathbf{0}$

Definition 3

a) $\mathbf{F}: R^n \to R^n$ is a P_0-function on a set $S \subseteq R^n$ if there exists an index *i* such that

$(x_i - y_i)\cdot(F_i(\mathbf{x})-F_i(\mathbf{y})) \geq 0 \quad \forall \mathbf{x},\mathbf{y} \in S$.

b) $\mathbf{F}: R^n \to R^n$ is a P -function on a set $S \subseteq R^n$ if there exists an index *i* such that

$(x_i - y_i)\cdot(F_i(\mathbf{x})-F_i(\mathbf{y})) > 0 \quad \forall \mathbf{x},\mathbf{y} \in S,\ \mathbf{x} \neq \mathbf{y}$.

c) $\mathbf{F}: R^n \to R^n$ is a uniform P -function on a set $S \subseteq R^n$ if there exists an index *i* such that

$(x_i - y_i)\cdot(F_i(\mathbf{x})-F_i(\mathbf{y})) \geq \mu\|\mathbf{x}-\mathbf{y}\|^2$ for some $\mu > 0 \ \forall \mathbf{x},\mathbf{y} \in S$.

Definition 4

a) an $n \times n$ matrix M is a P_0 *-matrix* if every principal minor of M is non-negative.

b) an $n \times n$ matrix M is a P *-matrix* if every principal minor of M is positive.

Definition 5 For a solution $\mathbf{x}^*$ of NCP(F), we introduce the following three index sets:

$$\alpha = \{\, i \mid x_i^* > 0,\ F_i(\mathbf{x}^*) = 0\},\quad \beta = \{\, i \mid x_i^* = F_i(\mathbf{x}^*) = 0\},\quad \gamma = \{\, i \mid x_i^* = 0,\ F_i(\mathbf{x}^*) > 0\}$$

For a vector $\mathbf{x} \in R^n$, $\mathbf{x}_\alpha$ represents the vector with elements $x_i, i \in \alpha$. Similarly, $\mathbf{F}_\alpha$ represents the vector function with component functions $F_i, i \in \alpha$. In addition, we denote ∇_α the differential operator with respect to $\mathbf{x}_\alpha$.

Definition 6 Let $\mathbf{x}^*$ be a solution of NCP(F).

a) A solution $\mathbf{x}^*$ is said to be *nondegenerate* if $\beta = \emptyset$.

b) $\mathbf{x}^*$ is said to be *b-regular* if , for an index set δ such that $\alpha \subseteq \delta \subseteq \alpha \cup \beta$, $\nabla_\delta \mathbf{F}_\delta(\mathbf{x}^*)$ is nonsingular. (see J.-S.Pang and S.A.Gabriel(1993))

c) $\mathbf{x}^*$ is said to be *R-regular* if $\nabla_\alpha \mathbf{F}_\alpha(\mathbf{x}^*)$ is nonsingular and its Schur complement in

$$\begin{bmatrix} \nabla_\alpha \mathbf{F}_\alpha(\mathbf{x}^*) & \nabla_\alpha \mathbf{F}_\beta(\mathbf{x}^*) \\ \nabla_\beta \mathbf{F}_\alpha(\mathbf{x}^*) & \nabla_\beta \mathbf{F}_\beta(\mathbf{x}^*) \end{bmatrix}$$

is a P -matrix. (see Robinson(1980))