The Cell Transmission Model, Newell's Cumulative Curves and Min-Plus Algebra

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(December 1, 2003)

1. Preliminaries – Daganzo's Cell Transmission Model

Suppose that the relationship between traffic flow, q, and density, k, in a homogeneous road section is of the following form:

$$q = \min\{vk,\ q_{\max},\ w(k_{jam} - k)\} \tag{1.1}$$

where v, q_{max}, w and k_{jam} are constants denoting the free-floe speed, the maximum flow (or capacity), the backward wave speed (with which disturbances propagate backward when traffic is congested), and the maximum ("jam") density, respectively (see **Figure 1**).

We assume that the road is divided into homogeneous sections ("cells") whose lengths equal the distance traveled by free-flowing traffic in one clock interval. The cells are numbered consecutively starting with the upstream end of the road from i = 0 to I. (see **Figure 2**) Each cell has the following two parameters: (1) the maximum number of vehicles that can be present in cell i at time t, $N_i(t)$, which is defined as the product of the cell's length and its jam density (k_{jam}), (2) the maximum number of vehicles that can be flow into cell i from time t to t+1 is denotes as $Q_i(t)$, which is the product of the clock interval and the cell's capacity (q_{max}).

Under the assumptions above, Daganzo(1994a,b) showed that the Lighthill-Whitham -Richard (LWR) kinematic wave equations for a single highway link can be approximated by a set of finite difference equations ("*cell-transmission model*"). In the model, the state of the system in the cell at time t is given by the number of vehicles contained in each cell, $n_i(t)$. The typical recursive relationship for the state variables,$\{n_i(t)\}$, is expressed as

$$n_i(t+1) = n_i(t) + y_i(t) - y_{i+1}(t) \tag{1.2a}$$

Here, $y_i(t)$ is the inflow to cell i in the time interval (t, $t+1$), which is given by

$$y_i(t) = \min\{n_{i-1}(t),\ Q_i(t),\ \delta[N_i(t) - n_i(t)]\}, \tag{1.2b}$$

where δ is defined as $\delta = 1$, if $n_{i-1}(t) \le Q_i(t)$ and $\delta = w/v$, if $n_{i-1}(t) > Q_i(t)$. In a nutshell, the cell-transmission model is the following system of difference equations with state variables $\{n_i(t)\}$:

$$\begin{aligned} n_i(t+1) = n_i(t) &+ \min\{n_{i-1}(t),\ Q_i(t),\ \delta[N_i(t) - n_i(t)]\} \\ &- \min\{n_i(t),\ Q_{i+1}(t),\ \delta[N_{i+1}(t) - n_{i+1}(t)]\} \end{aligned} \qquad (i = 1,2,\ldots,I\text{-}1) \tag{1.3}$$

2. Transforming the Model to "Linear Equations" in Min-Plus Algebra

The cell-transmission model defined in (1.2) is a seemingly complex system of *non-linear* equations. A simple transformation technique, however, allows us to obtain a more concise expression for the model; specifically, we shall show that the transformed model can be regarded as a system of simple "*linear equations*" from the standpoint of Min-Plus (Max -Plus) algebra.

(1) A Simple Variable Transformation

In order to see the essence of the idea, we first consider a simple case that $\delta = 1$, $N_i(t) = N$, and $Q_i(t) \to \infty$ in the original cell transmission model (1.2b):

$$y_i(t) = \min\{n_{i-1}(t),\ N - n_i(t)\} \tag{2.1}$$

Noticing that the cell transmission model in this setting is equivalent to the system of equations consisting of (1.2a) and

$$y_i(t) = \min\{n_{i-1}(t) - c,\ c - n_i(t)\}, \tag{2.2}$$

we introduce the following variable transformation:

$$n_i(t) = c + A_i(t) - A_{i+1}(t), \tag{2.3}$$

where $c \equiv N/2$. Applying this transformation to (2.2) yields,

$$\begin{aligned} y_i(t) &= \min\{A_{i-1}(t) - A_i(t),\ A_{i+1}(t) - A_i(t)\} \\ &= \min\{A_{i-1}(t),\ A_{i+1}(t)\} - A_i(t). \end{aligned} \tag{2.4}$$

Substituting this relationship and the transformation (2.3) into the conservation equation (1.2a), we see that the following equations hold:

$$A_i(t+1) - A_{i+1}(t+1) = \min\{A_{i-1}(t),\ A_{i+1}(t)\} - \min\{A_i(t), A_{i+2}(t)\},$$

$$A_{i-1}(t+1) - A_i(t+1) = \min\{A_{i-2}(t),\ A_i(t)\} - \min\{A_{i-1}(t), A_{i+1}(t)\}$$

$$\cdots$$

$$A_1(t+1) - A_2(t+1) = \min\{A_0(t), A_2(t)\} - \min\{A_1(t),\ A_3(t)\}. \tag{2.5}$$

Summing equations (2.5) over subscript *i*, we obtain

$$-A_{i+1}(t+1) - A_1(t+1) = -\min\{A_i(t),\ A_{i+2}(t)\} + \min\{A_0(t),\ A_2(t)\}. \tag{2.6}$$

In order for (2.6) to hold for all $i > 1$, the following equation should hold:

$$\underline{A_i(t+1) = \min\{A_{i-1}(t),\ A_{i+1}(t)\}}. \tag{2.7}$$

Thus, we found that non-linear equations (1.2a) and (2.1) (or (2.2)) reduce to a very concise expression, (2.7), in terms of the new state variables $\{A_i(t)\}$.

(2) An Interpretation from the Viewpoint of Min-Plus Algebra

We shall interpret the transformation process above from a view point of "Min-Plus algebra", which is defined as a set S equipped with two binary operations, $\oplus$ and $\otimes$:

$$\underline{x \oplus y \equiv \min(x, y)}\text{, and } \underline{x \otimes y \equiv x + y} \qquad \text{for all } \; x, y \in S$$

The cell transmission model, (1.2a) and (2.2), or equivalently,

$$n_i(t+1) = n_i(t) + \min\{n_{i-1}(t) - c,\; c - n_i(t)\} - \min\{n_i(t) - c,\; c - n_{i+1}(t)\}, \qquad (2.8)$$

can be represented in terms of the Min-Plus operations as

$$\begin{aligned} n_i(t+1) = n_i(t) \otimes \{[c^{-1} \otimes n_{i-1}(t)] \oplus [c \otimes n_i(t)^{-1}]\} \\ \otimes \{[c^{-1} \otimes n_i(t)] \oplus [c \otimes n_{i+1}(t)^{-1}]\}^{-1}, \end{aligned} \qquad (2.9)$$

and the transformation (2.3) as

$$n_i(t) = c \otimes A_i(t) \otimes A_{i+1}(t)^{-1}. \qquad (2.10)$$

Having learned that (2.8) reduces to (2.7) by the transformation (2.3), we see that (2.9) can be transformed into

$$A_i(t+1) = A_{i+1}(t) \oplus A_{i-1}(t). \qquad (2.11)$$

From the viewpoint of Min-Plus algebra, this transformation from (2.9) to (2.11) implies the remarkable fact that the *non-linear* system of equations reduces to a system of *linear* equations via (2.9). This "*linearity*" of the transformed equations (2.11) provides us useful insights for the system (although the linearity in Min-Plus algebra does not necessarily mean the same properties as those in linear algebra). For example, (1) if $\{B_i(t)\}$ and $\{C_i(t)\}$ satisfy (2.11), $\{B_i(t) \oplus C_i(t)\}$ is also the solution of (2.11); (2) if $\{A_i(t)\}$ satisfy (2.11) then $\{\alpha \otimes A_i(t)\}$ is also the solution of (2.11); (3) if appropriate boundary conditions are given, we can obtain an explicit solution. Indeed, we may write (2.11) as

$$\mathbf{A}(t+1) = \mathbf{M} \otimes \mathbf{A}(t), \qquad (2.12)$$

and we then obtain the solution of (2.11) for the initial condition that $\{A_i(1)\}$ is given:

$$\mathbf{A}(t+1) = \mathbf{M}^t \otimes \mathbf{A}(1), \qquad (2.13)$$

or
$$A_i(t+1) = \bigoplus_{j=i-t\,(++2)}^{i+t} A_j(1) = \min\{A_{i-t}(1), A_{i-t+2}(1), \cdots, A_{i+t-2}(1), A_{i+t}(1)\},$$

where the matrix $\mathbf{M}$ and the power of matrix $\mathbf{M}$ is defined as

$$\mathbf{M} \equiv \begin{bmatrix} o & e & & & & & \\ e & o & e & & & \mathbf{o} & \\ & e & o & e & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & e & o & e & \\ & \mathbf{o} & & & e & o & e \\ & & & & & e & o \end{bmatrix}, \quad \mathbf{M}^0 = \mathbf{E}, \qquad \mathbf{M}^k = \mathbf{M}^{k-1} \otimes \mathbf{M} \quad (k = 1, 2, \ldots).$$

The Min-Plus algebraic point of view further suggests an analogical correspondence to the differential equations theory; we observe a certain similarity between the transformation from (2.9) to (2.11) via (2.10) and the transformation of Berger's (shock wave) equation to a linear (heat) diffusion equation via Cole-Hopf transformation.

It is well known in the differential equation theory that the Berger's equation:

$$\frac{\partial m(t,x)}{\partial t} + 2m(t,x)\frac{\partial m(t,x)}{\partial x} = \frac{\partial^2 m(t,x)}{\partial x^2}, \tag{2.14}$$

reduces to a linear diffusion equation:

$$\frac{\partial A(t,x)}{\partial t} = \frac{\partial^2 A(t,x)}{\partial x^2}, \tag{2.15}$$

by the Cole-Hopf transformation defined as

$$m(x,t) = -\frac{\partial(\log A(x,t))}{\partial x} = -\frac{\partial A(x,t)}{\partial x}\frac{1}{A(x,t)}. \tag{2.16}$$

In order to see the correspondence to this transformation, let us first consider the following discrete (difference equation) approximation of the diffusion equation (2.15):

$$[A_j(t+\Delta t) - A_j(t)]/\Delta t = [A_{j+1}(t) - 2A_j(t) + A_{j-1}(t)]/(\Delta x)^2,$$

which can be simplified into

$$A_j(t+1) = (1/2)[A_{j+1}(t) + A_{j-1}(t)], \tag{2.17}$$

by setting $\alpha \equiv \Delta t/(\Delta x)^2$ =1/2. We then define a discretized Cole-Hopf transformation:

$$\begin{aligned} m_i(t)\Delta x &\equiv \ln A_i(t) - \ln A_{i+1}(t), \\ n_i(t) &\equiv c\cdot \exp(m_i(t)\Delta x) = c\,A_i(t)/A_{i+1}(t). \end{aligned} \tag{2.18}$$

Substituting (2.17) into the transformation equation (2.18), we have

$$\begin{aligned} n_i(t+1) &= c\frac{A_i(t+1)}{A_{i+1}(t+1)} = c\frac{A_{i+1}(t) + A_{i-1}(t)}{A_{i+2}(t) + A_i(t)}, \\ &= c\frac{A_i(t)}{A_{i+1}(t)}\frac{(A_{i+1}(t)/A_i(t)) + (A_{i-1}(t)/A_i(t))}{(A_{i+2}(t)/A_{i+1}(t)) + (A_i(t)/A_{i+1}(t))}, \end{aligned} \tag{2.19}$$

and rearranging the RHS of (2.19) yields the following equation called a finite difference Berger's equation (for more details about this, see Hirota (2000)):

$$n_i(t+1) = n_i(t)\frac{(c/n_i(t)) + (n_{i-1}(t)/c)}{(c/n_{i+1}(t)) + (n_i(t)/c)}. \tag{2.20}$$

Note here that replacing a pair of binary operations, $+$ and $\times$, in (2.20) with the Min-Plus operations, $\oplus$ and $\otimes$, we have exactly the cell-transmission model defined in (2.9). Similarly, it follows that the discretized diffusion equation (2.17) and the discretized Cole-Hopf transformation (2.18) correspond to (2.11) and (2.10), respectively. Thus, we see that the cell-transmission model can be regarded as an analogue (a finite difference version) of the second order linear differential equation when we employ the new state variables $\{A_i(t)\}$ and the Min-Plus algebraic operations for describing the evolution of the system.

3. A Cumulative Curve Representaion for the Cell-Transmission Model

We shall extend the analysis in **section 2** to the original cell transmission model. For this purpose, it is convenient to employ the following transformation:

$$n_i(t) = A_i(t) - A_{i+1}(t)\,, \tag{3.1a}$$

which is a slightly modified version of (2.3). Note that $A_{i+1}(t)$ in this transformation has a natural physical meaning: the cumulative number of vehicles arriving at the downstream-end of cell *i* (but still in cell *i*) by time *t* (see **Figure 3**); this is indeed a discrete approximation scheme of the relationship between the cumulative traffic counts, $A(x,t)$, and the traffic density, $k(x,t)$, for the continuous space-time pair (t, x):

$$k(x,t) = -\partial A(x,t)/\partial x\,. \tag{3.1b}$$

(1) The Cell-Transmission Model with a Single Wave Speed (ie. the case of $\delta = 1$)

Consider first the slightly extended version of the previous model (2.1):

$$n_i(t+1) = n_i(t) + y_i(t) - y_{i+1}(t)\,, \tag{3.2}$$

$$y_i(t) = \min\{n_{i-1}(t),\ N_i(t) - n_i(t)\}. \tag{3.3}$$

Applying the transformation (3.1a) to (3.3), we obtain

$$y_i(t) = -A_i(t) + \min\{A_{i-1}(t),\ N_i(t) + A_{i+1}(t)\}\,. \tag{3.4}$$

Substituting this and (3.1a) into the conservation equation (3.2) yields,

$$A_{i+1}(t+1) - A_i(t+1) = \min\{A_i(t), N_{i+1}(t) + A_{i+2}(t)\} - \min\{A_{i-1}(t),\ N_i(t) + A_{i+1}(t)\}\,, \quad \text{for all } i > 1. \tag{3.5}$$

In order for (3.5) to hold, we should have

$$\underline{A_i(t+1) = \min\{A_{i-1}(t),\ N_i(t) + A_{i+1}(t)\}} \qquad \text{for all } i > 1. \tag{3.6}$$

From the concise expression (3.6) for the cell transmission model, we see a few basic implications. First, the following relationship between $\{A_i(t)\}$ and $\{y_i(t)\}$ can be derived from (3.1a), (3.3) and (3.6):

$$y_i(t) = A_i(t+1) - A_i(t)\,, \tag{3.7a}$$

which corresponds to the relationship between the cumulative traffic count, $A(x,t)$, and the traffic flow, $q(x,t)$, for the continuous space-time pair (t, x):

$$q(x,t) = \partial A(x,t)/\partial t\,. \tag{3.7b}$$

Note that (3.7a) is not a definitional relationship here but is the result derived from the definitional relationships (3.1a), (3.3) and the state equation (3.6) in which the existence of

$A_i(t)$ guarantees a conservation of vehicles numbers. This in turn implies that an alternative derivation of the state equation (3.6) is given by simply substituting (3.1a) and (3.7a) into (3.3) without explicit consideration of (3.2). Such derivation of (3.6) corresponds to the fact that we can obtain the partial differential equation for describing the evolution of $A(x,t)$ in a continuous space-time system:

$$\frac{\partial A(x,t)}{\partial t} = \min\left\{ -v\,\frac{\partial A(x^-,t)}{\partial x},\ w\,(k_{\max} + \frac{\partial A(x^+,t)}{\partial x}) \right\}, \tag{3.8}$$

by substituting equations (3.1b) and (3.7b) (*ie*. the definitional relationships for $q(x,t)$, $k(x,t)$ and $A(x,t)$) into the following q-k relationship:

$$q(x,t) = \min\left\{ v\,k(x^-,t),\ w\,(k_{\max} - k(x^+,t)) \right\}. \tag{3.9}$$

Secondly, (3.6) can be read as a discrete version of Newell's "*lower envelop recipe*" (1993): we can obtain a "true" solution of the kinematic equations of LWR for the cumulative curve by taking the lower envelop of the multiple-valued solution derived from the conditions for a forward wave (the first term of the bracket in the RHS of (3.5)) and a backward wave (the second term). This is obvious from the fact that (3.6) is a discrete analogue of (3.8). Finally, we should note that (3.6) is just a system of *linear equations* from the view point of Min-Plus algebra; we may indeed write (3.5) as

$$\underline{A_i(t+1) = A_{i-1}(t) \oplus (N_i(t) \otimes A_{i+1}(t))}\,. \tag{3.10}$$

Note that "+" is represented as $\otimes$ in Min-Plus algebra (as defined in **section** 2 (2)).

The linearity of the cell-transmission model in Min-Plus algebra can be extended to the model with capacities (but assuming a single wave speed):

$$n_i(t+1) = n_i(t) + y_i(t) - y_{i+1}(t)\,, \tag{3.2}$$

$$y_i(t) = \min\{n_{i-1}(t),\ Q_i(t),\ N_i(t) - n_i(t)\}\,. \tag{3.11}$$

This fact can be easily verified by applying the transformation (3.1a); substituting (3.1a) and (3.7a) into (3.11), we see that the model above is equivalent to

$$\underline{A_i(t+1) = \min\{A_{i-1}(t),\ Q_i(t) + A_i(t),\ N_i(t) + A_{i+1}(t)\}}\,. \tag{3.12}$$

(3.12) implies that the lower envelop recipe still holds for the extended model (3.11), and that the model again reduces to a system of linear equations in terms of Max(Min)-Plus algebra:

$$\underline{A_i(t+1) = A_{i-1}(t) \oplus (Q_i(t) \otimes A_i(t)) \oplus (N_i(t) \otimes A_{i+1}(t))}\,. \tag{3.13}$$

Furthermore, it can be easily verified (see **Appendix 2**) that (3.13) can be regarded as an analogue of the linear diffusion equation in the sense that (2.11) is the analogue of the diffusion equation (2.15).

(2) The Cell-Transmission Model with Two Different Wave Speeds (ie. the case of $\delta < 1$)

Now, we are in a position to analyze the general cell transmission model (1.2):

$$n_i(t+1) = n_i(t) + y_i(t) - y_{i+1}(t)\,, \tag{3.2}$$

$$y_i(t) = \min\{n_{i-1}(t),\ Q_i(t),\ \delta[N_i(t) - n_i(t)]\}\,. \tag{3.21}$$

where $\delta = w/v$. In spite of the seemingly slight difference between this general model and the previous model (*ie*. $\delta = w/v < 1$ in (3.21) while $\delta = 1$ in (3.11)), a straightforward application of the previous transformation method to the general cell transmission model does not preserve the Min-Plus algebraic linearity of the system. Specifically, applying the transformation (3.1) to (3.21) yields

$$\begin{aligned} A_i(t+1) = \min\{&A_{i-1}(t),\ Q_i(t) + A_i(t),\\ &\delta N_i(t) + \delta A_{i+1}(t) + (1-\delta)A_i(t)\}, \end{aligned} \tag{3.22}$$

which is expressed in terms of Min-Plus algebraic operations as

$$\begin{aligned} A_i(t+1) = A_{i-1}(t) &\oplus [Q_i(t) \otimes A_i(t)]\\ &\oplus [(\delta N_i(t)) \otimes (\delta A_{i+1}(t)) \otimes ((1-\delta)A_i(t))]. \end{aligned} \tag{3.23}$$

This equation no longer has "linearity" in Min-Plus algebra (*ie*. the linearity is lost due to the third term in RHS bracket of (3.22) containing the "multiplication" of $A_i(t)$ and $A_{i+1}(t)$ (*ie*. $\delta A_{i+1}(t) \otimes (1-\delta)A_i(t)$).

The linearity of the model, however, can be recovered by exploiting the cumulative count based state equation, (3.8), for the continuous time-space version of LWR theory rather than the cell-transmission model. For a finite interval of space-time pair, $(\Delta x, \Delta t)$, the equation (3.8) can be approximated as

$$\frac{A(x,t+\Delta t) - A(x,t)}{\Delta t} = \min\left\{\begin{aligned} &-v\frac{A(x,t) - A(x-\Delta x,t)}{\Delta x},\\ &w\,k_{\max} + w\frac{A(x+\Delta x,t) - A(x,t)}{\Delta x} \end{aligned}\right\} \tag{3.24}$$

The finite difference equation (3.24) represents a propagation of either a forward wave (the first term in the RHS bracket) or a backward wave (the second term). This implies that

$$\Delta A(x^-,t) \equiv A(x-\Delta x, t+\Delta t) - A(x,t) = 0$$

holds whenever $\Delta x = -v \cdot \Delta t$ and the forward wave is effective in (3.24), because

$$\begin{aligned} \Delta A(x^-,t) &\approx \frac{A(x,t) - A(x-\Delta x,t)}{\Delta x}(-\Delta x) + \frac{A(x,t+\Delta t) - A(x,t)}{\Delta t}\Delta t\\ &= -[\Delta x + v\Delta t][A(x,t) - A(x-\Delta x,t)]/\Delta x. \end{aligned}$$

In other words, the following relationship holds along the forward characteristic curves:

$$\underline{A(x + v\cdot\Delta t, t+\Delta t) = A(x,t)} \tag{3.25}$$

Similarly, when the backward wave is effective,

$$\begin{aligned}\Delta A(x^{+},t) &\equiv A(x+\Delta x,t+\Delta t)-A(x,t)\\ &\approx \frac{A(x+\Delta x,t)-A(x,t)}{\Delta x}\Delta x+\frac{A(x,t+\Delta t)-A(x,t)}{\Delta t}\Delta t\\ &= wk_{\max}\Delta t+[\Delta x+w\Delta t][A(x+\Delta x,t)-A(x,t)]/\Delta x,\end{aligned}$$

which implies that $\Delta A(x^{+},t)=wk_{\max}$ holds whenever $\Delta x=-w\cdot\Delta t$; that is, we have

$$\underline{A(x-w\cdot\Delta t,t+\Delta t)=A(x,t)+wk_{\max}\Delta t} \tag{3.26}$$

along the backward characteristic curves. Note that (3.25) and (3.26) are exact and valid for any point (x, t), since we assumed a triangular (or trapezoidal) flow-density relationship (*ie*. the forward (backward) wave speed v (w) is constant independent of k, t and x).

In the cell transmission model with the cell length equal to $v\cdot\Delta t$ (*ie*. the distance traveled by forward wave speed in one clock interval), the relationship (3.25) for the forward wave is represented as

$$A_{i+1}(t+1)=A_i(t)\,,\quad A_i(t+1)=A_{i-1}(t)\,,\ \ldots \tag{3.27}$$

Similarly, the relationship (3.26) for the backward wave implies that

$$A_{i-1}(t+1)=A_i(t)+N_{i-1}(t)\,,\quad A_i(t+1)=A_{i+1}(t)+N_i(t)\,,\ldots \tag{3.28}$$

holds for the cell transmission model with the cell length equal to $w\cdot\Delta t$ (*ie*. the distance traveled by backward wave speed in one clock interval) (see **Figure 4**). Suppose here that the ration of forward wave speed to backward wave speed can be approximated by an appropriate natural number J, that is, $v/w=J$ (J = 2, 3, 4,…). Then, the relationship (3.25) for the forward wave becomes

$$A_{i+J}(t+1)=A_i(t)\,,\quad A_i(t+1)=A_{i-J}(t)\,,\ \ldots \tag{3.29}$$

Combining these equations, (3.28) and (3.30), as well as the capacity constraint in each cell into a single equation, we have the following expression for the general cell transmission model:

$$\underline{A_i(t+1)=\min\{\,A_{i-J}(t),\ Q_i(t)+A_i(t),\ N_i(t)+A_{i+1}(t)\,\}}, \tag{3.30}$$

where we assume that each cell length equals $w\cdot\Delta t$ and v/w can be approximated by a natural number J (see **Figure 4**). Note that the subscript shift J in the first term of the RHS bracket of (3.30) substitutes for the wave speed parameter δ (= w/v) in the original cell transmission model.

The equation (3.30) thus obtained is linear in Min-Plus algebra; it is indeed expressed as

$$\underline{A_i(t+1)=A_{i-J}(t)\oplus(Q_i(t)\otimes A_i(t))\oplus(N_i(t)\otimes A_{i+1}(t))}\,, \tag{3.31}$$

4. Applications of the Min-Plus Algebraic View

The "linear" expression (3.31) can be exploited to solve a certain class of kinematic wave problems with complicated boundary conditions (eg. Newell(1993)'s "three detector problem") that cannot be solved by a simple forward (with respect to time) computation of the cell transmission (recurrence) formula. To show this fact, we regard the discretized space–time plane constructed in **Section 3 (2)** as a "network" with an adjacency (cost) matrix **C** whose (u, v) element is given by

$$c(u,v) = \begin{cases} 0 & \textit{if } \text{grid } u = (i-J,t) \\ Q(u) & \textit{if } \text{grid } u = (i,t) \\ N(u) & \textit{if } \text{grid } u = (i+1,t) \\ +\infty & \text{otherwise} \end{cases} \qquad \text{for each grid } v = (i,\ t+1). \tag{3.34}$$

This means that each grid point on the discretized space–time plane corresponds to a vertex/node in the network (*ie*. a node in the network represents a space-time pair), and a node pair (u,v) in the network is connected by a directed link (with "link cost" $c(u,v)$) if the wave propagation from space-time point u to v is allowed (see **Figure 4**). We also denote by $A(u)$ the cumulative number of vehicles at a space-time point u (*ie*. $A(u) = A_i(t)$ if node u denotes a space-time point (i,t)).

For this network representation, it follows that the cumulative-number-based cell transmission model, (3.31), is expressed as

$$\mathbf{A} = \mathbf{C} \otimes \mathbf{A} \oplus \mathbf{b} \tag{3.32}$$

or

$$[\mathbf{I} - \mathbf{C}] \otimes \mathbf{A} = \mathbf{b} \tag{3.33}$$

where **A** is an unknown vector whose typical element denotes a cumulative number of vehicles, $A(u)$, at space-time point u, and the coefficient vector **b** is determined from given boundary conditions that determine the values of cumulative number of vehicles in a certain space-time domain.

As is well known in Min-Plus algebra, the solution of "linear equations" (3.33) can be found by the following matrix operations:

$$\mathbf{A} = [\mathbf{I} - \mathbf{C}]^{-1} \otimes \mathbf{b} = [\mathbf{I} \oplus \mathbf{C} \oplus \mathbf{C}^2 \oplus \cdots] \otimes \mathbf{b} \tag{3.35}$$

(Note here that (3.35) has a certain similarity with the solution:

$$\mathbf{A} = [\mathbf{I} - \mathbf{C}]^{-1} \mathbf{b} = [\mathbf{I} + \mathbf{C} + \mathbf{C}^2 + \cdots] \mathbf{b}$$

for the linear equations [**I** - **C**] **A** = **b** in ordinary matrix algebra). It is also well known that (3.32) means the optimality (DP) condition for a *minimum cost (shortest) path problem* in a network with adjacency matrix **C**, and that (3.35) corresponds to *Warshall-Floyd algorithm* for the shortest path problem. Thus, we see that kinematic wave problems with arbitrary complicated boundary conditions (that satisfy a certain solution existence conditions) can be solved efficiently either by a simple “linear” (in the sense of Min-Plus algebra) operations given in (3.35), or equivalently, by any algorithms for solving a shortest path problem.

References

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Appendix 1

We briefly summarize the elementary properties of Min-Plus algebra. It is easily verified that the operations $\oplus$ and $\otimes$ obey the following laws:

$x \oplus y = y \oplus x$, and $x \otimes y = y \otimes x$ (*ie*. commutative law),

$(x \oplus y) \oplus z = x \oplus (y \oplus z)$, and $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ (*ie*. associative law).

The operation $\oplus$ further satisfies $x \oplus x = x$ (*ie. idempotent law*) while $\otimes$ not. These operations also satisfies the *distributive law*: $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$.

The Min-plus algebra S contains a *zero element* o such that

$$x \otimes o = o \otimes x = o \quad \text{and} \quad x \oplus o = o \oplus x = x \qquad \text{for all} \quad x \in S,$$

and a *unit element* e such that

$$x \otimes e = e \otimes x = x \qquad \text{for all} \quad x \in S$$

It follows that $+\infty$ is the zero element and the number zero is the unit element (*ie.* $\underline{o = +\infty}$ and $\underline{e = 0}$). An inverse x^{-1} of a given element x with respect to $\otimes$ such that

$$x \otimes x^{-1} = x^{-1} \otimes x = e \qquad \text{for a given } x$$

is given by $\underline{x^{-1} = -x}$, while an inverse with respect to the operation $\oplus$ does not exist.

We may also consider "matrix operations" in this algebra. Let $M_n(S)$ be the set of all n by n matrices whose entry belong to S. we define two binary operations on $M_n(S)$ as

$$\mathbf{X} \otimes \mathbf{Y} = [\bigoplus_{k=1}^{n} (x_{ik} \otimes y_{kj})] \quad \text{and} \quad \mathbf{X} \oplus \mathbf{Y} = [x_{ij} \oplus y_{ij}],$$

for any matrices $\mathbf{X} = [x_{ij}]$ and $\mathbf{Y} = [y_{ij}]$ in $M_n(S)$, where we used the following operation for "summing" multiple elements { $x_i \in S$; i =1,2,…n}: $\bigoplus_{i=1}^{n} x_i \equiv x_1 \oplus x_2 \oplus ... \oplus x_n$ (Needless to say, this corresponds to $\sum_{i=1}^{N} x_i$ in ordinary algebra). A unit element $\mathbf{E}$ for the matrix multiplication such that

$$\mathbf{E} \otimes \mathbf{X} = \mathbf{X} \otimes \mathbf{E} = \mathbf{X} \quad \text{for all} \quad \mathbf{X} \in M_n(S)$$

is given by the n by n matrix whose diagonal entries are all e (unit element) and the other entries are all o (zero element).

It is convenient in some cases to consider "Max-Plus algebra", in which two binary operations, $\oplus$ and $\otimes$ are defined as $x \oplus y \equiv \max(x, y)$ and $x \otimes y \equiv x + y$ for all $x, y \in S$. The properties of this algebra is almost the same with those of Min-Plus algebra, except that the zero element is given as $\underline{o = -\infty}$. For more details of these algebras, see for example, Carre (1979) and Bacceli et al.(1992).

Appendix 2

To see the fact that (3.13) can be regarded as an analogue of the linear diffusion equation, consider the following variant of the diffusion equation:

$$\frac{\partial A(x,t)}{\partial t} = \frac{\partial^2 A(x,t)}{\partial x^2} + 2\beta(t,x)\frac{\partial A(x,t)}{\partial x} + (\beta(t,x)^2 - \alpha(t,x))\, A(x,t)\,, \quad (3.14)$$

which can be obtained from the basic diffusion equation of the form:

$$\partial a(t,x)/\partial t = \partial^2 a(t,x)/\partial x^2 \quad (3.15)$$

by the transformation $a(t,x) = e^{\alpha(t,x)\,t + \beta(t,x)\,x} A(t,x)$, and $A(0,x) = e^{-\beta(t,x)\,x} a(0,x)$. The standard difference approximation of (3.14) is

$$\frac{A_i(t+1) - A_i(t)}{\Delta t} = \frac{A_{i+1}(t) - 2A_i(t) + A_{i-1}(t)}{(\Delta x)^2} + \frac{2\beta_i \cdot (A_{i+1}(t) - A_i(t))}{\Delta x} - (\beta_i^2 - \alpha_i)A_i(t)\,, \quad (3.16)$$

Setting $\Delta t/(\Delta x)^2 = 1$, $N_i(t) \equiv 1 + 2\beta_i(t)\Delta x$ and $Q_i(t) \equiv (\alpha_i(t) - \beta_i(t)^2)\Delta t - 2\beta_i(t)\Delta x - 1$ in (3.16), we have

$$A_i(t+1) = A_{i-1}(t) + Q_i(t)\,A_i(t) + N_i(t)\,A_{i+1}(t)\,, \quad (3.17)$$

which has exactly the same form as (3.13) when "+" and "$\times$" are read as the Min-Plus algebraic operations.

Appendix 3

Although Daganzo (1994) defined δ as $\delta = 1$, if $n_{i-1}(t) \le Q_i(t)$ and $\delta = w/v$, otherwise (*ie*. if $n_{i-1}(t) > Q_i(t)$), the definition is equivalent to simply setting $\delta = w/v$. This can be verified as follows: when the former definition of δ is employed,

$$\begin{aligned} &\min\{Q_i(t),\ n_{i-1}(t),\ \delta[N_i(t) - n_i(t)]\} \\ &= \min\{\min(n_{i-1}(t), N_i(t) - n_i(t)),\ \min(Q_i(t),\ (w/v)[N_i(t) - n_i(t)])\} \\ &= \min\{n_{i-1}(t), Q_i(t), N_i(t) - n_i(t),\ (w/v)[N_i(t) - n_i(t)]\} \end{aligned}$$

holds because if $n_{i-1}(t) \le Q_i(t)$, then $\delta = 1$ and

$$\min\{Q_i(t),\ n_{i-1}(t),\ \delta[N_i(t) - n_i(t)]\} = \min\{n_{i-1}(t),\ N_i(t) - n_i(t)\}\,,$$

otherwise (*ie*. $n_{i-1}(t) > Q_i(t)$), $\delta = w/v$ and

$$\min\{Q_i(t),\ n_{i-1}(t),\ \delta[N_i(t) - n_i(t)]\} = \min\{Q_i(t),\ (w/v)[N_i(t) - n_i(t)]\}\,.$$

Since the backward wave speed is assumed to be slower than the forward wave speed (*ie*. $w/v < 1$), it follows

$$\begin{aligned} &\min\{n_{i-1}(t), Q_i(t), N_i(t) - n_i(t),\ (w/v)[N_i(t) - n_i(t)]\} \\ &= \min\{n_{i-1}(t), Q_i(t), (w/v)[N_i(t) - n_i(t)]\}, \end{aligned}$$

which is equivalent to setting $\delta = w/v$ in (3.21).

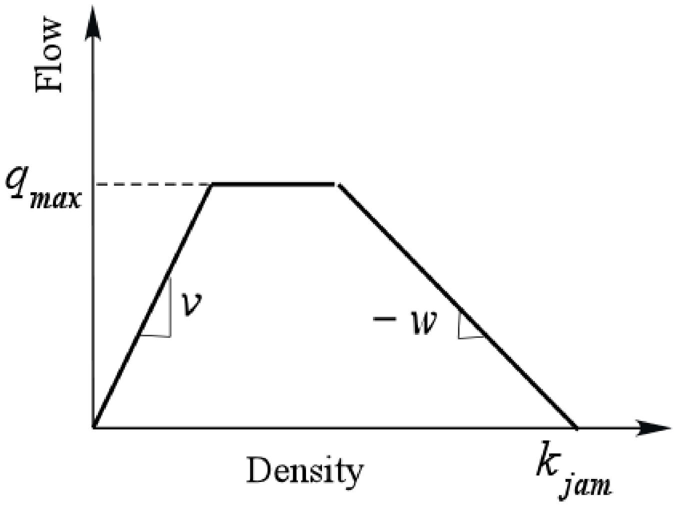

Figure 1 Flow-density relationship for the cell-transmission model

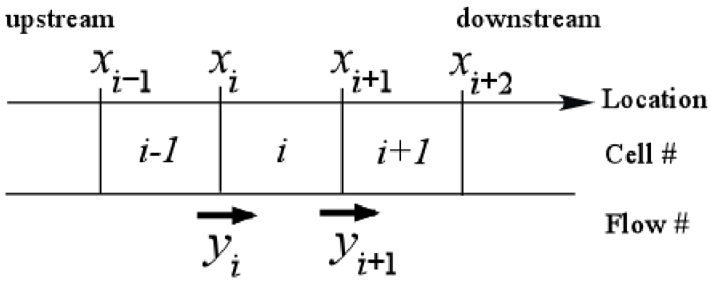

Figure 2 Locations and sections (“cells”) in a road.

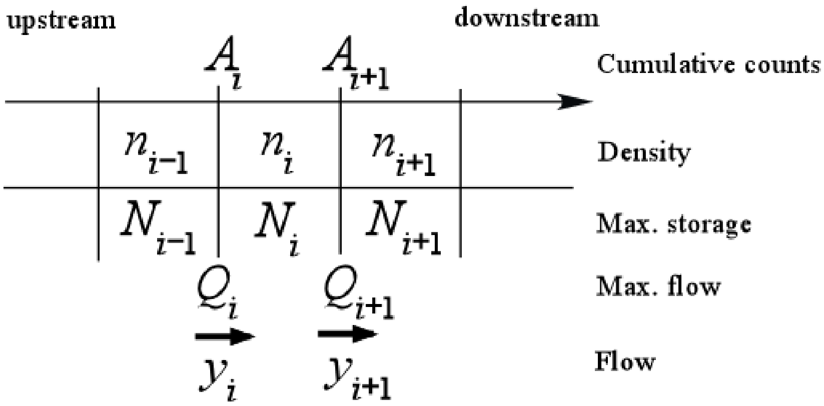

Figure 3 State variables and parameters in each cell

$n_i(t) = A_i(t) - A_{i+1}(t) = k(x_i \sim x_{i+1};t)\Delta x$, $N_i(t) = k_{jam}(x_i \sim x_{i+1};t)\Delta x$,

$y_i(t) = A_i(t+1) - A_i(t) = q(x_i,t)\Delta t$, $Q_i(t) = q_{\max}(x_i,t)\Delta t$

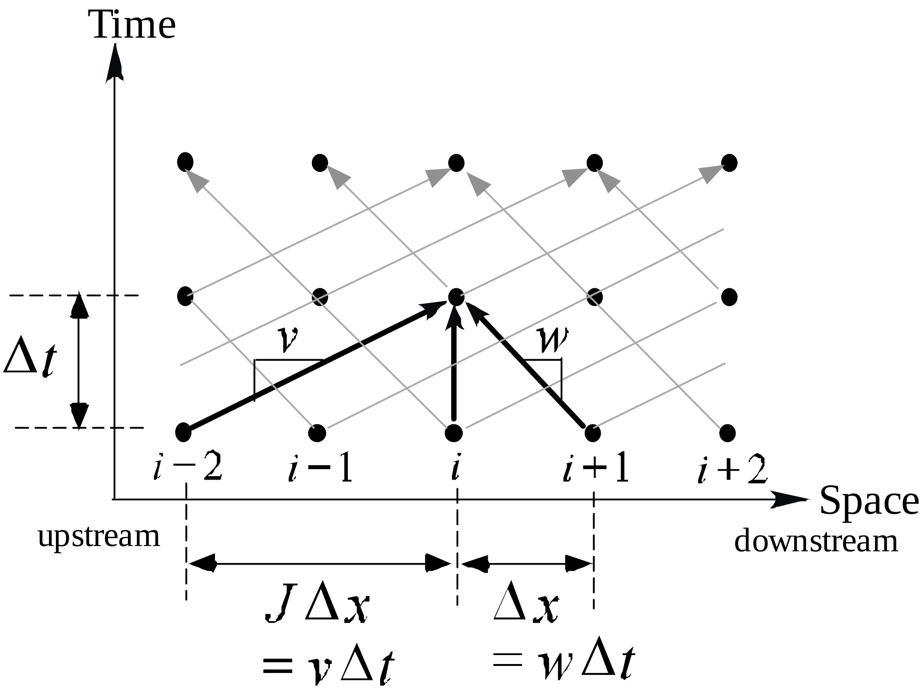

Figure 4 Characteristic curves on a space-time grid for the cumulative count based cell transmission model (the case of $J = v / w = 2$)