A Generalized Complementarity Approach to Solving Real Option Problems ⋆

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Abstract

This article provides a unified framework for analyzing a wide variety of real option problems. These problems include the frequently studied, simple real option problems, as described in Dixit and Pindyck (1994) for example, but also problems with more complicated and realistic assumptions. We reveal that all the real option problems belonging to the more general class considered in this study are described by the same mathematical structure, which can be solved by applying a computational algorithm developed in the field of mathematical programming. More specifically, all of the present real option problems can be directly solved by reformulating their optimality condition as a dynamical system of generalized linear complementarity problems (GLCPs). This enables us to develop an efficient and robust algorithm for solving a broad range of real option problems in a unified manner, exploiting recent advances in the theory of complementarity problems.

Key words: Real options; Generalized complementarity problem; Smoothing function-based algorithm

JEL classification: C61; C63; D92; E22; G31

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1 Introduction

Following the pioneering works of Brennan and Schwartz (1985) and McDonald and Siegel (1985), an enormous number of studies have been carried out on the theory of real option and its applications. For example, the comprehensive works of Dixit and Pindyck (1994), Schwartz and Trigeorgis (2001), and the references therein. However, most of these studies focused only on the "plain vanilla" option model. In this model, the option to invest is "killed" when investment is undertaken, and no future investment can take place. While this type of model has rightly received considerable attention in the literature, more complex situations exist for which this model is not appropriate.

A few studies have, however, been undertaken treating real option problems involving more complicated and realistic situations. These works can be roughly classified into two categories —– entry-exit options and time-to-build options. In the former category, Dixit (1989) and Dixit and Pindyck (1994, chap.7) have pioneered the expansion of the basic concept of irreversible real options to a model with partially reversible investment, hereafter referred to as cyclic real option problems. In order to solve these problems, they adopted a conventional approach, known as the value-matching and the smooth-pasting (or high-contact) conditions (hereafter referred to as VM-SP). This approach is particularly useful if closed-form analytical solutions are sought, because it reduces a real option problem to a tractable system of nonlinear equations. However, it is difficult to apply the VM-SP approach to quantitative analysis of real option problems involving more practical situations (e.g., a finite-horizon model in which each of the state variables follows a generalized Ito diffusion process).

In the latter category, Majd and Pindyck (1987) formulated and analyzed an irreversible investment problem with lags. They also used the VM-SP approach to analyze their model, which causes a serious omission of an essential optimality condition, as pointed out by Milne and Whalley (2000). There also exist other studies of time-to-build options, for example, Bar-Ilan and Strange (1996) and Bar-Ilan et al. (2002). These studies analyzed a hybrid model of a time-to-build option and an entry-exit option, by reformulating it as a quasi-variational inequality problem. Although this approach is potentially quite general, they also used the VM-SP approach to solve the problem by imposing several restrictive assumptions. To the best of our knowledge, there have been no studies that systematically analyze various real option problems under realistic and generalized situations.

The present paper provides a unified framework for analyzing a wide variety of real option problems, taking into account the practical aspects of real-world investments. The main contribution of this article is to reveal that all the real option problems belonging to the more general class considered here are described by the same mathematical structure, and that this structure can be solved by applying a computational algorithm developed in the field of mathematical programming. More precisely, all of these apparently different real option problems can be universally reformulated as a system of generalized linear complementarity problems (GLCPs). This enables us to develop an efficient and robust algorithm for solving a wide variety of real option problems, —– some of which cannot be solved analytically or numerically using existing approaches —– in a unified manner, exploiting recent advances in the theory of complementarity problems.

The GLCP approach here can be regarded as a natural extension of the LCP (or, the *variational inequality*) approach, which is introduced by Jaillet et at. (1990) as an equivalent representation of “plain vanilla” American option problems. The LCP approach is the currently most favored method for pricing vanilla options, being superior to other existing methods in terms of accuracy and efficiency. More precisely, for other more primitive existing methods such as the binomial approximation methods, the option prices and the optimal strategies obtained from a discretized model may not converge to their continuous counterpart if the underlying state variable does not follow a geometric Brownian motion but a more generalized diffusion process. See Appendix A for a more detailed review of the existing numerical methods and their limitations. Despite its advantages, the LCP approach is not directly applicable to the more complicated real option problems discussed in the present paper, because these real option problems reduce to systems of GLCPs rather than standard LCPs. We thus should develop a new numerical method to solve these systems of GLCPs.

To demonstrate the proposed framework and solution algorithm, we consider two applications, each of which is a generalized version of the frequently studied real option problems described above: a) the entry-exit option model with a finite horizon, in which the market price of the output follows a generalized Ito process; and b) the time-to-build option model, in which the firm’s instantaneous profit is defined as an arbitrary function of the market price of the output following a generalized Ito process. Note that hitherto there has been no systematic method for solving these problems in general. The traditional VM-SP approach can only be used to solve *simplified* forms of these problems: the entry-exit option in an infinite time horizon with a state variable following a geometric Brownian motion (Dixit and Pindyck, 1994); and the time-to-build option with a linear instantaneous profit function and a state variable whose dynamics is formulated as a geometric Brownian motion (Majd and Pindyck, 1987; Milne and Whalley, 2000). It is also worthwhile to note that the present framework is not only applicable to a variety of existing real option problems, but also to a generalized version of multi-options (e.g., Trigeorgis, 1991, 1993; Kulatilaka, 1995), which consist of many sub-options interlinking each other, as discussed in Section 7.

The structure of the present paper is as follows: Sections 2 and 3 formulate an entry-exit option problem and a time-to-build option problem, respectively. The optimality conditions of each problem are then reformulated as a system of infinite-dimensional GLCPs. Section 4 shows that the system is decomposed with respect to time under an appropriate discrete framework. This enables us to reduce the real option problems to the problem of successively solving a sequence of subproblems, each of which is formulated as a finite-dimensional GLCP. Section 5 provides an efficient and robust algorithm for solving the subproblems. In Section 6 the present method for real option problems is applied to derive numerical solutions to several test problems. The efficiency of the proposed algorithm is clearly demonstrated. Section 7 concludes this paper.

We first introduce some notation: $\mathcal{R}_+^{\mathrm{N}}$ and $\mathcal{R}_{++}^{\mathrm{N}}$ respectively denote the nonnegative orthant and the positive orthant in an N-dimensional real space $\mathcal{R}^{\mathrm{N}}$, where, for the purpose of notational simplicity, $\mathcal{R}_+, \mathcal{R}_{++}$, and $\mathcal{R}$ are also used when $\mathrm{N} = 1$.

2 Cyclic Entry-Exit Option

This section deals with an entry-exit option. The model described in the present section is similar to that of Dixit (1989), except that we assume a finite horizon $[0, T]$, and further assume that the state variable follows a (generalized) Ito process. In what follows, we first define the present entry-exit option problem, and then reformulate the optimality condition of the problem as a system of infinite-dimensional generalized linear complementarity problems (GLCPs).

2.1 The Model

Suppose that, at every time, t, in a certain operation period $t \in [0, T]$, a firm is in one of two states: in the market and active (denoted by $m(t) = 1$) or outside the market and idle (denoted by $m(t) = 0$). A firm that is outside the market is able to enter the market by incurring a fixed cost C_E, whereas a firm that is in the market is able to leave the market by incurring a fixed cost C_Q. It is assumed that this entry-exit cycle may be repeated any number of times during the operation period. We therefore refer to this type of option as a “cyclic” entry-exit option.

When the firm is in the market, it produces a certain unit flow of output. The amount of the instantaneous profit per unit time at time t is defined by a known function $\pi_1(t, P(t))$, where $P(t)$ is the market price of output at t, which evolves exogenously over time following an Ito process:

$$\mathrm{d}P(t) = \alpha(t, P)\,\mathrm{d}t + \sigma(t, P)\,\mathrm{d}W(t), \quad P(0) = P_0, \tag{1}$$

where $W(t)$ is a standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$; Ω is the state space, $\mathcal{F}$ is the filtration of Ω, and $\mathcal{P}$ is the probability measure on $(\Omega, \mathcal{F})$. On the other hand, when the firm is outside the market, regardless of the market price, it neither incurs profits nor costs. The firm's instantaneous profit is thus denoted by $\pi_0(t, P(t)) = 0$ [1] .

The firm is assumed to decide its entry-exit (or idle-active) strategy in order to maximize the expected net present value of all future profit streams. For the interval to the end of the operating period $[t, T]$, the net present value of the profit streams subject to an entry-exit strategy $\{m(s)\}_t^T \triangleq \{m(s)|s \in [t, T]\}$ is defined as

$$\phi_A(t, T; m(\cdot)) \triangleq \int_t^T e^{-\rho(s-t)}\pi_{m(s)}(s, P(s))\,\mathrm{d}s - \sum_{k\geq 1} e^{-\rho(\tau_E^k - t)}C_E - \sum_{k\geq 1} e^{-\rho(\tau_Q^k - t)}C_Q + e^{-\rho(T-t)}\Pi_{m(T)}(P(T)), \quad (2)$$

where the discount rate, ρ, is a given constant. In Eq. (2), the four terms on the right-hand side respectively represent the net present value of i) the instantaneous profits, ii) the entry (investment) cost, iii) the quit (abandonment) cost, and iv) lump-sum profit obtained at the end of the operation. In the second and third terms, τ_E^k is the kth entry time and τ_Q^k is the kth exit time, respectively. In the last term, $\Pi_m(P)$ is a known function of P, which represents a lump-sum profit produced at the expiration date of the operation $t = T$, when the market price is $P(T) = P$ and the firm's state is $m(T) = m$. The problem of the firm's entry-exit decision during the operation period $[0, T]$ is formulated as the following stochastic control problem,

$$\text{[P-A]} \quad \max_{\{m(t)\}_0^T} \mathbb{E}\left[\phi_A(0, T; m(\cdot))\middle|P(0) = P_0, m(0) = 0\right].$$

The present setting reduces to the traditional model of Dixit (1989) when we assume: i) an infinite horizon (i.e., $T \to \infty$); ii) a geometric Brownian motion for the state variable (i.e., $\alpha(t, P) \triangleq \alpha P$ and $\sigma(t, P) \triangleq \sigma P$); and iii) a linear instantaneous profit function $\pi_1(t, P) \triangleq P - w$, where α, σ, and w are given positive constants. Although the traditional VM-SP approach may be used to solve this simplified problem, it is no longer applicable to our problem formulated above (i.e., a finite time horizon with a state variable following a Ito process and an arbitrary instantaneous profit function). It should be further stressed that no systematic method has been developed for solving the generalized problem, and thus as yet no numerical solutions have been obtained.

2.2 The Optimality Condition

This section derives the optimality condition of the [P-A]. In our framework, the optimality condition is represented as a single system of generalized linear com-

[1] This assumption can be relaxed without any change in the mathematical structure of the present framework.

plementarity problems (GLCPs), rather than a combination of the value-matching conditions and the smooth-pasting conditions.

We first define the value function of [P-A] as

$$V_0(t,P) \triangleq \max_{\{m(s)\}_t^T} \mathbb{E}\left[\phi_A(t,T;m(\cdot))\middle|P(t)=P, m(t)=0\right], \tag{3}$$

$$V_1(t,P) \triangleq \max_{\{m(s)\}_t^T} \mathbb{E}\left[\phi_A(t,T;m(\cdot))\middle|P(t)=P, m(t)=1\right]. \tag{4}$$

The former, V_0, represents the value function when the market price is $P(t) = P$ and the firm is outside the market (i.e., $m(t) = 0$) at time t, whereas the latter, V_1, is the value function when the firm is in the market (i.e., $m(t) = 1$) at time t. We can interpret $V_0(t,P)$ to be the value of the firm outside the market, whereas $V_1(t,P)$ is the value of the firm in the market. In what follows, we first derive the optimality condition (of the firm in the market) for leaving the market, and then the optimality condition (of the firm outside the market) for entering the market. Since the entry-exit cycle can be repeated infinitely, these two optimality conditions must be combined, so, finally, we formulate the combined optimality conditions as an infinite-dimensional GLCP.

Let us suppose that the firm is in the market when the market price is $P(t) = P$ at $t \in [0,T]$. By applying the dynamic programming (DP) principle, we see that the firm takes one of two actions: either it exits the market incurring exit cost C_Q, or defers exiting for at least a certain time Δ. From the definition, the value function should satisfy

$$V_1(t,P) \geq \int_t^{t+\Delta} e^{-\rho(s-t)}\pi_1(s,P(s))\,\mathrm{d}s + e^{-\rho\Delta}\mathbb{E}\left[V_1(t,P)+\Delta V_1(t,P)|P(t)=P, m(t)=1\right], \tag{5}$$

where $\Delta V_1(t,P) \triangleq \int_t^{t+\Delta} \mathrm{d}V_1(s)$, and this relation holds with equality when the firm chooses to postpone its exit. Taking $\Delta \to +0$ and using Ito's lemma, it must be true that

$$F_1(t,P) \triangleq -\mathcal{L}V_1(t,P) - \pi_1(t,P) \geq 0, \tag{6}$$

where $\mathcal{L}$ is an infinitesimal generator (partial differential operator), defined as

$$\mathcal{L} \triangleq \frac{\partial}{\partial t} + \alpha(t,P)\frac{\partial}{\partial P} + \frac{1}{2}\{\sigma(t,P)\}^2\frac{\partial^2}{\partial P^2} - \rho. \tag{7}$$

The value function also should satisfy

$$V_1(t,P) \geq V_0(t,P) - C_Q, \tag{8}$$

or, equivalently,

$$G_1(t,P) \triangleq V_1(t,P) - V_0(t,P) + C_Q \geq 0, \tag{9}$$

where $V_0(t,P)$ is the value of the firm outside the market at time t, as defined by Eq. (3). If the firm chooses to exit the market, relation (9) holds with equality. Since one of the two actions must be optimal, either Eq. (6) or (9) holds as an equality. Hence, the firm's optimal exit strategy (or more precisely, the optimality condition for leaving the market for a firm in the market) can be rewritten as

$$F_1(t, P) \cdot G_1(t, P) = 0, \quad F_1(t, P) \geq 0, \quad G_1(t, P) \geq 0. \tag{10}$$

Similarly, let us suppose that the firm is outside the market (i.e., $m(t) = 0$) when the market price is $P(t) = P$ at $t \in [0, T]$. By applying the DP principle, we see that the firm takes one of two actions: either it enters the market by paying the investment cost C_E, or suspends this investment. If the firm chooses to remain outside the market, then the following inequality, which is naturally derived from the definition of the value function, must hold with equality,

$$F_0(t, P) \triangleq -\mathcal{L}V_0(t, P) \geq 0. \tag{11}$$

where $\mathcal{L}$ is the infinitesimal generator (partial differential operator) defined by Eq. (7). If the firm, on the other hand, chooses to enter the market, the following inequality must instead hold with equality,

$$V_0(t, P) \geq V_1(t, P) - C_E, \tag{12}$$

or equivalently,

$$G_0(t, P) \triangleq V_0(t, P) - V_1(t, P) + C_E \geq 0. \tag{13}$$

Hence, the firm's optimal entry strategy (or more precisely, the optimality condition for entering the market for a firm outside the market) can be formulated as

$$F_0(t, P) \cdot G_0(t, P) = 0, \quad F_0(t, P) \geq 0, \quad G_0(t, P) \geq 0. \tag{14}$$

It should be noted that the firm can repeat its entry-exit cycle any number of times. This implies that the value of the firm outside the market $V_0(t, P)$ is required for calculating the value of the firm in the market $V_1(t, P)$ in Eq. (10), and vice versa. Therefore, conditions (10) and (14) should hold simultaneously, and the values $V_0(t, P)$ and $V_1(t, P)$ should be obtained as a solution of the following system of GLCPs:

[GLCP-A] Find $\{[V_0(t, P), V_1(t, P)] | (t, P) \in [0, T] \times \mathcal{R}_+\}$ such that

$$\begin{cases} F_0(t, P) \cdot G_0(t, P) = 0, \quad F_0(t, P) \geq 0, \quad G_0(t, P) \geq 0, \\ F_1(t, P) \cdot G_1(t, P) = 0, \quad F_1(t, P) \geq 0, \quad G_1(t, P) \geq 0 \end{cases}, \forall (t, P) \in [0, T] \times \mathcal{R}_+.$$

The terminal condition at $t = T$ is given by

$$V_{m(T)}(T, P(T)) = \Pi_{m(T)}(P(T)), \quad \forall m(T) \in \{0, 1\}, \forall P(T) \in \mathcal{R}_+. \tag{15}$$

3 Time-to-Build Option

3.1 *The Model*

This section formulates a time-to-build option using the same framework and notation as Majd and Pindyck (1987), except for a slight generalization of the state variable process. Let us consider a building project given by the construction a factory that cannot be completed in a single day. The amount of capital required to complete the factory, $\bar{K}$, is known. We assume that there is a maximum rate of investment, k, and that the investment is also irreversible; therefore the rate of investment, $I(t)$, has the constraint

$$0 \leq I(t) \leq k. \tag{16}$$

We also assume that previously installed capital does not decay. Let $K(t)$ denote the remaining expenditure required at time t. The dynamics of $K(t)$ is then given by

$$\mathrm{d}K(t) = -I(t)\,\mathrm{d}t, \quad K(0) = \bar{K}, \quad K(\tau) = 0, \tag{17}$$

where τ is the completion date of the factory, which is determined by the investment policy and the remaining capital.

Upon completion ($K(\tau) = 0$), the factory commences the production of a certain amount of output whose market price $P(t)$ evolves stochastically. The completed factory is assumed to possess an infinite life $[\tau, \infty)$ and be capable of generating cash flow streams $\{\pi(P(t))|t \in [\tau, \infty)\}$. Thus, the value of the factory at the completion date is the expected net present value of all future cash flow streams during its infinite life span, which is defined as

$$\Pi(P) \triangleq \mathbb{E}\left[\int_\tau^\infty e^{-\rho(s-\tau)}\pi(P(s))\,\mathrm{d}s \middle| P(\tau) = P\right]. \tag{18}$$

The dynamics of the market price of output is modeled by a stationary Ito process

$$\mathrm{d}P(t) \triangleq \alpha(P)\,\mathrm{d}t + \sigma(P)\,\mathrm{d}W(t), \quad P(0) = P_0, \tag{19}$$

where $\alpha : \mathcal{R}_+ \to \mathcal{R}$, $\sigma : \mathcal{R}_+ \to \mathcal{R}_+$ are known functions and $W(t)$ is a standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

Let us consider a certain construction period $[t, \tau]$ and an investment strategy $\{I(s)\}_t^\tau \triangleq \{I(s)|s \in [t, \tau)\}$. The net present value of all future cash flow streams is then defined as

$$\phi_B(t, \tau; I(\cdot)) \triangleq -\int_t^\tau e^{-\rho(s-t)}I(s)\,\mathrm{d}s + e^{-\rho(\tau-t)}\Pi(P(\tau)), \tag{20}$$

where the discount rate, ρ, is a given constant. We assume that the manager of the building project intends to maximize the expected net present value of all future cash flow streams of the project by choosing an investment strategy $\{I(t)|t \in [0, \tau]\}$. This objective is formulated as

$$\text{[P-B]} \quad \max_{\{I(t)\}_0^\tau} \mathbb{E}\left[\phi_B(0, \tau; I(\cdot))\middle| K(0) = \bar{K}, P(0) = P_0\right]. \tag{21}$$

Note that the traditional VM-SP approach cannot be used to solve this problem, since we assume that the state variable $P(t)$ follows a generalized Ito process (19) and define the payoff function $\Pi(P)$ as an arbitrary function. The present problem reduces to that of Majd and Pindyck (1987) if we set $\Pi(P) \triangleq P$ and the market price process is given by a geometric Brownian motion (i.e., $\alpha(P) \triangleq \alpha P$ and $\sigma(P) \triangleq \sigma P$ with α and σ given constants). This *simplified* problem can be solved by using either the VM-SP approach or our approach. However, a naïve application of the VM-SP may result in a serious omission of essential optimality conditions, which can be avoided by using our approach. This will be discussed in the following section.

3.2 The Optimality Condition

We show that the optimality condition of [P-B] can also be reformulated as a GLCP, in a similar way to the entry-exit option in Section 2. First, we define the value function of [P-B] by

$$V(K, P) \triangleq \max_{\{I(s)\}_t^\tau} \mathbb{E}\left[\phi_B(t, \tau; I(\cdot))\middle| K(t) = K, P(t) = P\right], \tag{22}$$

where the remaining expenditure is $K(t) = K$ and the market price of the output is $P(t) = P$. It should be noted that the assumption of an infinite horizon implies that the basic characteristics and the solutions of [P-B] —– the optimal investment policy and the value function —– do not depend on time *per se*. We have therefore omitted the time description hereafter.

By applying the DP principle to Eq. (22) and using Ito's lemma, we obtain the following HJB (Hamilton-Jacobi-Bellman) equation.

$$\mathcal{D}V(K, P) + \max_{0 \le I \le k} \left\{ -\frac{\partial V(K, P)}{\partial K} - 1 \right\} I = 0, \tag{23}$$

where $\mathcal{D}$ is an infinitesimal generator (ordinary differential operator) defined by

$$\mathcal{D} \triangleq \alpha(P)\frac{\partial}{\partial P} + \frac{1}{2}\sigma^2(P)\frac{\partial^2}{\partial P^2} - \rho. \tag{24}$$

Since the HJB equation (23) is linear with respect to the control variable $I(t)$, we obtain the optimal investment policy as “bang-bang”, or,

$$\begin{cases} I = 0, & \text{if } -\dfrac{\partial V(K,P)}{\partial K} - 1 < 0, \\ I = k, & \text{if } -\dfrac{\partial V(K,P)}{\partial K} - 1 \geq 0, \end{cases} \tag{25}$$

that is, either to invest at the maximum rate k if the marginal benefit of investing $(\partial V/\partial K)$ is higher than the marginal cost -1, or not to invest at all.

As shown below, this optimal strategy (25) reduces to a system of infinite-dimensional GLCPs. If the investment takes place at the maximum rate, it must be true that

$$\begin{cases} F(K,P) \triangleq -\mathcal{D}V(K,P) - \left(-\dfrac{\partial V(K,P)}{\partial K} - 1\right)k = 0, \\ G(K,P) \triangleq -\mathcal{D}V(K,P) \geq 0, \end{cases} \tag{26}$$

where the inequality in the second line is obtained from $F(K,P) = 0$ and $-\frac{\partial V(K,P)}{\partial K} - 1 \geq 0$. On the other hand, when there is no investment, it must be true that

$$\begin{cases} G(K,P) = 0, \\ F(K,P) > 0, \end{cases} \tag{27}$$

where the inequality is derived from $G(K,P) = 0$ and $-\frac{\partial V(K,P)}{\partial K} - 1 < 0$. The optimality conditions (26) and (27) are summarized as the following GLCP:

[GLCP-B] Find $\{V(K,P) | (K,P) \in [0,\bar{K}] \times \mathcal{R}_+\}$ such that

$$F(K,P) \cdot G(K,P) = 0, \quad F(K,P) \geq 0, \quad G(K,P) \geq 0, \quad (K,P) \in [0,\bar{K}] \times \mathcal{R}_+.$$

The terminal condition held at the completion date τ is given by

$$V(0, P(\tau)) = \Pi(P(\tau)), \quad \forall P(\tau) \in \mathcal{R}_+. \tag{28}$$

It should be noted that the proposed GLCP approach does not involve explicit free boundaries (or what are often referred to as the triggers, the thresholds, or the cutoff values). In our approach, the GLCP is derived from the original optimality condition for the value function (i.e., the HJB equation), and the free-boundary is obtained as a consequence of solving the GLCP, and the value-matching and the smooth-pasting conditions are “automatically” satisfied. This implies that the GLCP approach has an advantage in that it avoids the serious omission of essential optimality conditions that can occur if using a free-boundary and the value-matching and smooth-pasting conditions a priori, as highlighted by Milne and Whalley (2000).

4 Decomposition into Finite Dimensional GLCPs

The previous two sections have shown that the two apparently different options have a common mathematical structure, namely an infinite-dimensional GLCP. This section shows that both these infinite-dimensional GLCP systems, [GLCP-A] and [GLCP-B], can be decomposed into a series of subproblems by using the DP principle with respect to time. Next, it is shown that each of the real option problems discussed in Sections 2 and 3 reduces to the problem of successively solving the subproblems, each of which is formulated as a finite-dimensional GLCP, in an appropriate discretized framework.

4.1 Entry-Exit Option

Let us suppose a sufficiently large subspace $[P_{\min}, P_{\max}]$ in the state (the market price) space $\mathcal{R}_+$, and a discrete grid in the time-state space $[0, T] \times [P_{\min}, P_{\max}]$ with increments Δt and ΔP . We denote each point of the grid by $(t^i, P^j) \triangleq (i\Delta t, P_{\min} + j\Delta P)$, where the indices $i = 0, 1, \cdots, \mathrm{I}$ and $j = 0, 1, \cdots, \mathrm{J}, \mathrm{J} + 1$ describe the locations of the point with respect to time and state, respectively. In this framework, we denote an arbitrary function $X_m : [0, T] \times \mathcal{R}_+ \to \mathcal{R}\, \forall m \in \{0, 1\}$ at a grid point (t^i, P^j) by $X_m^{i,j}$, and let $\boldsymbol{X}_m^i \triangleq \{X_m^{i,1}, \cdots, X_m^{i,\mathrm{J}}\}^\top$ denote the value of $X_m(t, P)$ at time t^i. We also use the following 2J-dimensional vectors

$$\boldsymbol{V}^i \triangleq \begin{bmatrix} \boldsymbol{V}_0^i \\ \boldsymbol{V}_1^i \end{bmatrix}, \qquad \boldsymbol{F}^i \triangleq \begin{bmatrix} \boldsymbol{F}_0^i \\ \boldsymbol{F}_1^i \end{bmatrix}, \qquad \boldsymbol{G}^i \triangleq \begin{bmatrix} \boldsymbol{G}_0^i \\ \boldsymbol{G}_1^i \end{bmatrix}.$$

In the present discretized framework, the infinitesimal generator $\mathcal{L}$ can be approximated by using an appropriate finite-difference scheme (see, e.g., Jaillet et al., 1990) as follows:

$$\mathcal{L}V_m(t^i, P) \approx \boldsymbol{L}^i \boldsymbol{V}_m^i + \boldsymbol{M}^i \boldsymbol{V}_m^{i+1}, \quad \forall m = 0, 1, \forall i = 0, 1, \cdots, \mathrm{I} - 1. \tag{29}$$

where $\boldsymbol{L}^i$ and $\boldsymbol{M}^i$ are $\mathrm{J} \times \mathrm{J}$ square matrices determined by the market price process (1). Then the problem [GLCP-A] can be rewritten as a set of finite-dimensional GLCPs:

[GLCP-A(D)] Find $\{\boldsymbol{V}^i \in \mathcal{R}^{2\mathrm{J}} | i = 0, 1, \cdots, \mathrm{I}\}$ such that

$$\boldsymbol{F}^i(\boldsymbol{V}^i, \boldsymbol{V}^{i+1}) \cdot \boldsymbol{G}(\boldsymbol{V}^i) = 0, \quad \boldsymbol{F}^i(\boldsymbol{V}^i, \boldsymbol{V}^{i+1}) \geq \boldsymbol{0}, \quad \boldsymbol{G}(\boldsymbol{V}^i) \geq \boldsymbol{0}, \quad \forall i = 0, 1, \cdots, \mathrm{I} - 1.$$

where

$$\boldsymbol{F}^i(\boldsymbol{V}^i, \boldsymbol{V}^{i+1}) \triangleq \begin{bmatrix} -\boldsymbol{L}^i\boldsymbol{V}_0^i - \boldsymbol{M}^i\boldsymbol{V}_0^{i+1} - \boldsymbol{0} \\ -\boldsymbol{L}^i\boldsymbol{V}_1^i - \boldsymbol{M}^i\boldsymbol{V}_1^{i+1} - \boldsymbol{\pi}^i \end{bmatrix}, \qquad \boldsymbol{G}(\boldsymbol{V}^i) \triangleq \begin{bmatrix} \boldsymbol{V}_0^i - \boldsymbol{V}_1^i + \mathbf{1}_{\mathrm{J}}C_E \\ \boldsymbol{V}_1^i - \boldsymbol{V}_0^i + \mathbf{1}_{\mathrm{J}}C_Q \end{bmatrix},$$

and $\mathbf{1}_{\mathrm{J}}$ is a J-dimensional column vector with all elements equal to 1. The terminal condition (15) is also rewritten as

$$\boldsymbol{V}^{\mathrm{I}} = \boldsymbol{\Pi}, \tag{30}$$

where $\boldsymbol{\Pi} \triangleq \left\{\Pi_0^1, \cdots, \Pi_0^{\mathrm{J}}, \Pi_1^1, \cdots, \Pi_1^{\mathrm{J}}\right\}^\top$ is a 2J-dimensional column vector with components $\Pi_m^j \triangleq \Pi_m(P^j)$.

Problem [GLCP-A(D)] consists of subproblems [GLCP-A^i] ($i = 0, 1, \cdots, \mathrm{I} - 1$), corresponding to the time grid. Note that the ith subproblem can be solved if a solution of the one-step-ahead subproblem, $\boldsymbol{V}^{i+1}$, is given. Hence, [GLCP-A(D)] reduces to the problem of successively solving the subproblems in the following procedure:

[Algo-A]

Step 0 Set $\boldsymbol{V}^{\mathrm{I}} := \boldsymbol{\Pi}$, and $i := \mathrm{I} - 1$.
Step 1 If $i < 0$, then STOP.
Step 2 Obtain $\boldsymbol{V}^i$ as the solution of [GLCP-A^i] by regarding $\boldsymbol{V}^{i+1}$ as a given constant.
Step 3 Set $i := i - 1$ and return to Step 1.

The algorithm for solving each subproblem [GLCP-A^i] will be discussed in Section 5.

A few remarks are in order: First, in the case of an infinite horizon, the system of GLCPs [GLCP-A(D)] reduces to a single GLCP, which can be readily solved as a single subproblem of [GLCP-A(D)]. In other words, the present approach is universally applicable to both the infinite-horizon and finite-horizon cases without any essential modification. It should be emphasized that real option problems in an infinite horizon can not be solved by the existing, more primitive methods, such as the binomial approximation method, since, in contrast to the present method, in these methods backward induction is unavoidable.

Second, if [P-A] was a plain vanilla American option rather than the cyclic option, [GLCP-A(D)] reduces to a system of *standard* LCPs as shown in Appendix C. For the system of standard LCPs, several numerical solution methods have been developed, and their accuracy and efficiency have been compared with those of the other existing methods (see Huang and Pang, 1998; Dempster and Hutton, 1999; and Coleman et al. 2002); this comparison is further discussed in Appendix A. However, in the case of a cyclic option, [GLCP-A(D)] does not reduce to a system of standard LCP, and the above existing methods for solving LCP systems are no longer applicable. We thus must develop a new numerical solution method to solve the GLCPs, as discussed in Section 5.

4.2 Time-to-Build Option

We take a sufficiently large subspace $[P_{\min}, P_{\max}] \subset \mathcal{R}_+$ and a discrete grid in the space corresponding to the remaining expenditure and the market price $[0, \overline{K}] \times [P_{\min}, P_{\max}]$ with increments ΔK and ΔP. We denote each point of the grid by $(K^i, P^j) \triangleq (\overline{K} - i\Delta K, P_{\min} + j\Delta P)$, where the indices $i = 0, 1, \cdots, \mathrm{I}$ and $j = 0, 1, \cdots, \mathrm{J}, \mathrm{J}{+}1$ characterize the locations of the points with respect to the remaining expenditure and the market price, respectively. Similar to the previous discretized framework, we denote the value of an arbitrary function $X : [0, \overline{K}] \times \mathcal{R}_+ \to \mathcal{R}$ at the grid point (K^i, P^j) by $X^{i,j}$. We also use a J-dimensional column vector, $\boldsymbol{V}^i \triangleq \{V^{i,1}, \cdots, V^{i,\mathrm{J}}\}^\top$ to denote the value function when the remaining expenditure is K^i.

In the present discretized framework, the differential operator $\mathcal{D}$ in Eqs. (26) and (27) can be approximated as

$$\mathcal{D}V(K^i, P) \approx \boldsymbol{D}\boldsymbol{V}^i, \tag{31}$$

$$\mathcal{D}V(K^i, P) - \frac{\partial V(K^i, P)}{\partial K} k \approx \boldsymbol{L}\boldsymbol{V}^i + \boldsymbol{M}\boldsymbol{V}^{i+1}, \tag{32}$$

where $\boldsymbol{D}$, $\boldsymbol{L}$, and $\boldsymbol{M}$ are $\mathrm{J} \times \mathrm{J}$ square matrices determined by the remaining expenditure process (17) and the market price process (19). Problem [GLCP-B] can then be expressed as a set of finite-dimensional GLCPs:

[GLCP-B(D)] Find $\{\boldsymbol{V}^i \in \mathcal{R}^{\mathrm{J}} | i = 0, 1, \cdots, \mathrm{I}\}$ such that

$$\boldsymbol{F}(\boldsymbol{V}^i, \boldsymbol{V}^{i+1}) \cdot \boldsymbol{G}(\boldsymbol{V}^i) = 0, \quad \boldsymbol{F}(\boldsymbol{V}^i, \boldsymbol{V}^{i+1}) \geq \boldsymbol{0}, \quad \boldsymbol{G}(\boldsymbol{V}^i) \geq \boldsymbol{0}, \quad \forall i = 0, \cdots, \mathrm{I} - 1,$$

where

$$\boldsymbol{F}(\boldsymbol{V}^i, \boldsymbol{V}^{i+1}) \triangleq -\boldsymbol{L}\boldsymbol{V}^i - \boldsymbol{M}\boldsymbol{V}^{i+1} + \boldsymbol{1}_{\mathrm{J}} k, \qquad \boldsymbol{G}(\boldsymbol{V}^i) \triangleq -\boldsymbol{D}\boldsymbol{V}^i, \tag{33}$$

and $\boldsymbol{1}_{\mathrm{J}}$ is a J-dimensional column with all elements equal to 1. The terminal condition (28) is also rewritten as

$$\boldsymbol{V}^{\mathrm{I}} = \boldsymbol{\Pi}, \tag{34}$$

where $\boldsymbol{\Pi} \triangleq \left\{\Pi^1, \cdots, \Pi^{\mathrm{J}}\right\}^\top$ is a Jth-order column vector whose jth element $\Pi^j \triangleq \Pi(P^j)$ is the value of the completed factory defined by Eq. (18).

We denote the subproblems of [GLCP-B(D)] for the ith grid of the remaining expenditure by [GLCP-B^i]. Analogous to the entry-exit option discussed above, the ith subproblem [GLCP-B^i] can be solved given a solution, $\boldsymbol{V}^{i+1}$, of the one-step-ahead subproblem. Hence, [GLCP-B(D)] reduces to a problem of successively solving the subproblems from $i = \mathrm{I} - 1$ to $i = 0$ in the same procedure as [Algo-A]; where the subproblem in step 2 is, of course, replaced by [GLCP-B^i].

5 Algorithm for Solving the Subproblem

The previous section showed that each of the real option problems [P-A] and [P-B] reduces to a problem of successively solving subproblems, each of which is formulated as a finite-dimensional GLCP. This section provides the algorithm for solving the subproblems [GLCP-A^i] and [GLCP-B^i]. In this section, we simply express both subproblems [GLCP-A^i] and [GLCP-B^i] as

$$\text{[GLCP]} \quad \text{Find } \boldsymbol{V} \in \mathcal{R}^{\mathrm{J}} \text{ such that } \boldsymbol{F}(\boldsymbol{V}) \cdot \boldsymbol{G}(\boldsymbol{V}) = 0, \quad \boldsymbol{F}(\boldsymbol{V}) \geq \boldsymbol{0}, \quad \boldsymbol{G}(\boldsymbol{V}) \geq \boldsymbol{0},$$

where $\boldsymbol{F}, \boldsymbol{G} : \mathcal{R}^{\mathrm{J}} \to \mathcal{R}^{\mathrm{J}}$ are known maps with jth elements $F^j(\boldsymbol{V})$ and $G^j(\boldsymbol{V})$, respectively. This type of finite-dimensional GLCP was first introduced by Cottle and Dantzig (1970), and many solution algorithms have been developed in various fields, including mathematical programming, control theory, engineering, and economics. For recent works, see Ferris and Pang (1997), Peng (1999), Qi and Liao (1999), Peng and Lin (1999), and the references therein.

In order to solve the subproblem [GLCP], we use the smoothing function approach developed by Peng (1999) , Qi and Liao (1999), and Peng and Lin (1999). This approach is not only a state-of-the-art technique, but is also well-suited to our problems from the view point of efficiency, as is discussed later.

In the smoothing function approach, one solves the following equivalent system of nonlinear equations rather than the original problem,

$$\boldsymbol{\mathcal{H}}(\boldsymbol{V}) \triangleq \min.\{\boldsymbol{F}(\boldsymbol{V}), \boldsymbol{G}(\boldsymbol{V})\} = \boldsymbol{0}, \tag{35}$$

where $\min.\{\boldsymbol{F}, \boldsymbol{G}\}$ is a vector operator whose jth element is defined as $\min.\left\{F^j, G^j\right\}$. Note that the system of equations, $\boldsymbol{\mathcal{H}}(\boldsymbol{V}) = \boldsymbol{0}$, cannot be solved by naïve methods, since $\boldsymbol{\mathcal{H}}(\boldsymbol{V})$ is undifferentiable, even if either $\boldsymbol{F}(\boldsymbol{V})$ or $\boldsymbol{G}(\boldsymbol{V})$ is affine.

In order to overcome difficulties caused by the undifferentiability of $\boldsymbol{\mathcal{H}}$, the key idea of the smoothing approach is to transform the original problem [GLCP] into a system of smooth equations via a so-called smoothing function $\boldsymbol{H}(\boldsymbol{V}, \xi)$ with jth component

$$H^j(\boldsymbol{V}, \xi) \triangleq -\xi \ln \left\{ \exp\left[-\frac{F^j(\boldsymbol{V})}{\xi} \right] + \exp\left[-\frac{G^j(\boldsymbol{V})}{\xi} \right] \right\}, \tag{36}$$

where $\xi \geq 0$ is referred to as the *smoothing parameter*. The type of function expressed in Eq. (36) is also known as an expected minimum cost (or a LOG-sum function) for a LOGIT model in random utility theory (e.g., McFadden, 1974; Daganzo, 1979; Ben-Akiva and Lerman, 1985). In this literature, it is known that the smoothing function has two desirable properties for developing an efficient algorithm: First, $\boldsymbol{H}(\boldsymbol{V}, +0) \triangleq \lim_{\xi \to +0} \boldsymbol{H}(\boldsymbol{V}, \xi) = \boldsymbol{\mathcal{H}}(\boldsymbol{V})$. In other words, the solution of the smooth equations system $\boldsymbol{H}(\boldsymbol{V}, \xi) = \boldsymbol{0}$ is equivalent to the solution of GLCP $\boldsymbol{\mathcal{H}}(\boldsymbol{V}) = \boldsymbol{0}$ in the limit as $\xi \to 0$; second, $H^j(\boldsymbol{V}, \xi)$ is a continuously differentiable function of $\boldsymbol{V}$ for all $\xi > 0$. The former property ensures that the present algorithm provides a good approximation to the solution of [GLCP], whereas the latter property is exploited to guarantee the efficiency of the algorithm.

The smoothing approach-based algorithm generates a solution set to the smooth equations system, forming a path $\{(\boldsymbol{V}, \xi) | \boldsymbol{H}(\boldsymbol{V}, \xi) = \boldsymbol{0}\}$ as the parameter ξ tends to zero. This path is usually referred to as the *smoothing path*. Let $\xi^{(k)}$ denote the smoothing parameter in the kth iteration, and $\boldsymbol{V}^{(k)}$ be a solution of the corresponding smooth equation $\boldsymbol{H}(\boldsymbol{V}, \xi^{(k)}) = \boldsymbol{0}$. In that case, we are able to summarize the procedure for generating the smoothing path as follows:

[Algo-GLCP]

Step 0. Choose $\xi^{(1)} \in \mathcal{R}_+$. Set $k := 1$;
Step 1. If $\boldsymbol{\mathcal{H}}(\boldsymbol{V}^{(k)}) = \boldsymbol{0}$ stop; $\boldsymbol{V}^{(k)}$ is the approximate solution of the GLCP;
Step 2. Solve the smooth equations system $\boldsymbol{H}(\boldsymbol{V}^{(k)}, \xi^{(k)}) = \boldsymbol{0}$;
Step 3. Choose the next smoothing parameter $\xi^{(k+1)} \in [0, \xi^{(k)})$;
Step 4. Set $k := k + 1$, return to Step1.

It is easy to verify that any accumulation point of the smoothing path $\{(\boldsymbol{V}^{(k)}, \xi^{(k)})\}$ generated by [Algo-GLCP] is the solution of the [GLCP], $(\boldsymbol{V}^*, +0)$, since the first property of the smoothing function $\boldsymbol{H}(\boldsymbol{V}, +0) = \boldsymbol{\mathcal{H}}(\boldsymbol{V})$ and the condition applicable on the smoothing parameters, $\xi^{(k)} > \xi^{(k+1)} \geq 0$, is satisfied. The global convergence of [Algo-GLCP] has been established (e.g., Peng and Lin, 1999): Any smoothing path $\{(\boldsymbol{V}^{(k)}, \xi^{(k)})\}$ generated by [Algo-GLCP] converges to $(\boldsymbol{V}^*, +0)$ globally, when i) $\nabla \boldsymbol{H}^{(k)} \triangleq \nabla \boldsymbol{H}(\boldsymbol{V}^{(k)}, \xi^{(k)})$ is non-singular, and ii) the norm of $(\nabla \boldsymbol{H}^{(k)})^{-1}$ is finite for all k. Since both the conditions are naturally satisfied in our framework, the smoothing path $\{(\boldsymbol{V}^{(k)}, \xi^{(k)})\}$ globally converges to the solution of [GLCP].

We conclude this section with a discussion of the efficiency of [Algo-GLCP]. We first emphasize that the smooth equations system $\boldsymbol{H}(\boldsymbol{V}, \xi) = \boldsymbol{0}$ can be solved by any Newton-type method thanks to the continuous differentiability of $H^j(\boldsymbol{V}, \xi)$, the second property of the smoothing function. In the kth iteration of [Algo-GLCP], the Newton direction $\boldsymbol{d}^{(k)}$ is calculated as a solution of the following system of linear equations, given the kth temporal solution, $(\boldsymbol{V}^{(k)}, \xi^{(k)})$,

$$\nabla \boldsymbol{H}(\boldsymbol{V}^{(k)}, \xi^{(k)}) \boldsymbol{d}^{(k)} + \boldsymbol{H}(\boldsymbol{V}^{(k)}, \xi^{(k)}) = \boldsymbol{0}, \tag{37}$$

where $\nabla \boldsymbol{H}(\boldsymbol{V}^{(k)}, \xi^{(k)})$ is the Jacobian of the smoothing function $\boldsymbol{H}$ evaluated at $(\boldsymbol{V}^{(k)}, \xi^{(k)})$.

The efficiency of [Algo-GLCP] depends on whether or not the system of linear equations (37) can be solved (i.e., the Newton direction $\boldsymbol{d}^{(k)}$ can be evaluated) efficiently. Fortunately, efficiency is guaranteed by the following two desirable properties of our problems: First, the evaluation of $\boldsymbol{H}(\boldsymbol{V}^{(k)}, \xi^{(k)})$ requires neither a time-consuming computational task nor a prohibitive storage for any large J, since the evaluation of the maps $\boldsymbol{F}(\cdot)$ and $\boldsymbol{G}(\cdot)$ reduces to a simple calculation of block matrices, $\nabla \boldsymbol{F}$ and $\nabla \boldsymbol{G}$, with blocks consisting of sparse matrices $\boldsymbol{L}$, $\boldsymbol{M}$, $\boldsymbol{D}$ and identity matrices.

Second, the Jacobian of the smoothing function, $\nabla \boldsymbol{H}(\boldsymbol{V}, \xi)$, is sparse, since it can be rewritten as a linear combination of sparse matrices $\nabla \boldsymbol{F}$ and $\nabla \boldsymbol{G}$,

$$\nabla \boldsymbol{H}(\boldsymbol{V}, \xi) \triangleq \boldsymbol{\Lambda}(\boldsymbol{V}, \xi) \nabla \boldsymbol{F}(\boldsymbol{V}) + [\boldsymbol{I} - \boldsymbol{\Lambda}(\boldsymbol{V}, \xi)] \nabla \boldsymbol{G}(\boldsymbol{V}), \tag{38}$$

where $\boldsymbol{I}$ is the Jth-order identity matrix and $\boldsymbol{\Lambda} : \mathcal{R}^{\mathrm{J}} \times \mathcal{R}_{+} \to \mathcal{R}^{\mathrm{J}\times\mathrm{J}}$ is a diagonal matrix whose (j, j) element is

$$\Lambda^{j}(\boldsymbol{V}, \xi) \triangleq \frac{1}{1 + \exp\left[\{F^{j}(\boldsymbol{V}) - G^{j}(\boldsymbol{V})\}/\xi\right]}. \tag{39}$$

The relation between the smoothing function $\boldsymbol{H}(\boldsymbol{V}, \xi)$ and its Jacobian in Eq. (38) can be naturally derived from the expected minimum cost and the (binomial) Logit choice probabilities. The sparsity of $\nabla \boldsymbol{H}$ enables us to solve the linear system of equations (37) efficiently, by using iterative algorithms such as the Gauss-Seidel algorithm or the Successive Over-Relaxation (SOR) algorithm, which only require storage of the non-zero elements of $\nabla \boldsymbol{H}(\boldsymbol{V}^{(k)}, \xi^{(k)})$. Therefore, [Algo-GLCP] is efficient for sufficiently large-scale subproblems.

6 Numerical Examples

This section describes some numerical examples of the entry-exit problem [P-A] and the time-to-build option problem [P-B]. In order to examine the accuracy of the present method, we compare our results with those of previous studies by using the same setting and parameter values as those studies. We also demonstrate the efficiency and robustness of the algorithm [Algo-GLCP]. Rather than considering complicated specialized problems in the general class of problems that the present method makes possible to solve, we consider several illustrative test problems that demonstrate that the present method can solve different types of real option problems in a unified manner.

6.1 Entry-Exit Option

For the entry-exit option, we use the same settings as Dixit and Pindyck (1994, Chap. 7), except that we assume a finite-horizon. We first assume that the market price follows a geometric Brownian motion,

$$\mathrm{d}P(t) = \alpha P(t)\,\mathrm{d}t + \sigma P(t)\,\mathrm{d}W(t), \quad P(0) = P_0. \tag{40}$$

Next, we assume that a firm in the market produces at a constant rate of 1 unit of output per unit of time, whose marginal cost is a given constant w. In other words, the instantaneous profit per unit time will be

$$\pi(t, P(t)) \triangleq P(t) - w. \tag{41}$$

The base case parameters in our numerical experiments are as follows: The duration of the operation period is $T = 30$ years and the annual discount rate is $\rho = 0.04$. The annual appreciation and the annual volatility of the market price are $\alpha = 0$ [2] and $\sigma = 0.2$, respectively. The marginal cost of production is $w = 0.8$, the lump-sum required to enter the market is $C_E = 20$, and the exit cost $C_Q = 2$. For the sake of simplicity, the lump-sum profit at the expiry date is assumed to be $\Pi(P) = 0$.

It is well-known that the simplified entry-exit option problem in an infinite time horizon has two thresholds that determine the optimal entry-exit strategy (e.g., Dixit and Pindyck, 1994). In other words, a firm outside the market enters the market when the market price $P(t)$ exceeds one threshold, P_E, whereas a firm in the market leaves the market when the price falls below another threshold, P_Q, at any moment $t \in [0, \infty)$. In contrast to the infinite-horizon model of Dixit and Pindyck, the thresholds in our finite-horizon model are time varying. Figure 1 shows these thresholds as functions of time. We are able to observe that both thresholds asymptotically approach those of the time-invariant case, $P^*_E = 1.34$ and $P^*_Q = 0.55$ (shown as dotted lines), when a sufficiently long period of the operation period remains. At the end of the operation period $t = T$, the entry threshold $P_E(t)$ goes to infinity, while the exit threshold $P_Q(t)$ goes to 0. It should be noted that this property cannot be observed in the analysis of models with an infinite-horizon.

Next, we demonstrate the robustness and efficiency of the present algorithm [Algo-GLCP]. Our parameters are: The numbers of grid points used in the discretization are I = J = 300, and the numerical parameters of the algorithm are $\delta_1 = 0.9, \delta_2 = 0.85, \delta_3 = 0.001, \xi^{(0)} = 1$ and $\epsilon_0 = 1.0 \times 10^{-7}$. Figure 2 illustrates the number of iterations required to solve each subproblem. In this figure, the horizontal axis, i, represents the index of the subproblem of [GLCP-A(D)], and the vertical axis represents the number of iterations that are required to solve the corresponding

[2] The setting $\alpha = 0$ in our framework corresponds to the setting $\mu = 0.04$ and $r = 0.04$ in Dixit and Pindyck (1994, Chap, 7).

subproblems. We observe that the algorithm [Algo-GLCP] is capable of finding a solution to each of the subproblems within a moderate number of iterations, ranging from 3 to 19 iterations with an average of 12.9. It may also be seen that the 299th subproblem requires the least, and the 282nd subproblem the largest number of iterations.

Figure 3 shows the convergence pattern for both [GLCP-A^{299}] and [GLCP-A^{282}]. In this figure, the horizontal axis, k, represents the iteration index, and the vertical axis represents the discrepancy between the kth temporal solution $\boldsymbol{V}^{(k)}$ and the exact solution $\boldsymbol{V}^*$ of the subproblem. It should be noted that the vertical axis has a *log-scale*: this figure shows that the numerical solutions obtained from algorithm [Algo-GLCP] converge to the corresponding exact solutions at an extremely fast rate, even for the subproblem that requires the largest number of iterations to be solved.

6.2 Time-to-Build Option

In the case of the time-to-build option problem [P-B], we use the same setting as that used by Milne and Whalley (2000). We assume that the market price of the output follows a geometric Brownian motion

$$\mathrm{d}P(t) = \alpha P(t)\,\mathrm{d}t + \sigma P(t)\,\mathrm{d}W(t), \quad P(0) = P_0, \tag{42}$$

and define the payoff for completing the building as $\Pi(P) = P$. The base case parameters used in our numerical experiments are as follows. The initial amount of capital required to complete the factory is $\overline{K} = 6$ and the maximum investment rate per unit time is $k = 1$. The annual discount rate is $\rho = 0.02$. The annual appreciation and the annual volatility of the market price are $\alpha = 0$ and $\sigma = 0.4$, respectively.

It is known that the time-to-build option problem has a critical value, $P^*(K)$, which provides the optimal investment strategy (see, e.g., Milne and Whalley, 2000). In other words, the investment should take place at the maximum rate when the market price of the output $P(t)$ is higher than the threshold, while no investment should be undertaken when the market price is lower than this threshold. Figure 4 shows the evolution of these thresholds, $P^*(K)$, as a function of the remaining expenditure K. Figure 4 also shows the other two thresholds, $P^0(K)$ and $P^1(K)$, which give alternative investment strategies in Milne and Whalley (2000). $P^0(K)$ is the investment threshold which emerges from the “naïve” net present value (NPV) rule, whereas $P^1(K)$ is the investment threshold in the case that the project, once commenced, must be carried through to completion. We see that the commencement-suspension threshold $P^*(K)$ is always higher than the NPV threshold $P^0(K)$, but is always lower than the commencement threshold for undeferrable investment $P^1(K)$. These results are consistent with those of Milne and Whalley (2000).

Finally, we show robustness and efficiency of the present method for the time-to-build option problem [P-B]. In what follows, the numbers of grid points used in the discretization are $I = J = 400$, and the parameters of the algorithm are the same as those in the previous section. Figure 5 shows the number of iterations required for each subproblem. In this figure, the horizontal axis, i, represents the index of each subproblem of [GLCP-B(D)], and the vertical axis represents the number of iterations. From this figure it may be seen that each of the subproblems is solved within a moderate numbers of iterations, ranging from 5 to 12 iterations with an average of 9.6.

Figure 6 shows the convergence patterns for both [GLCP-B^4] and [GLCP-B^{399}]; the former required the smallest number of iterations to reach convergence, while the latter required the the largest number of iterations to reach convergence. It may be noted that the temporal solution $\boldsymbol{V}^{(k)}$ quite rapidly converges to the exact solution $\boldsymbol{V}^*$ for each problem. It can be concluded from these results that [Algo-GLCP] is efficient for both the entry-exit option problem [P-A] and the time-to-build option problem [P-B].

7 Concluding Remarks

This article provides a unified approach to analyzing a wide variety of real option problems, taking into account the practical aspects of real-world investments, such as a finite-horizon in which each of the state variables follows a generalized Ito process. We first formulated a generalized version of two typical real option problems —– the entry-exit option problems and the time-to-build option problems. We then revealed that all the real option problems belonging to the more general class considered in this study are described by the same mathematical structure, which can be solved by applying a computational algorithm developed in the field of mathematical programming. In more precise terms, we found that the Bellman optimality conditions of these apparently different real option problems can be universally reduced to a dynamical system of generalized linear complementarity problems (GLCPs). This enables us to develop an efficient and robust algorithm for solving these real option problems in a unified manner, exploiting recent advances in the theory of complementarity problems.

The present computational method for solving real option problems has four main desirable properties: First, the present approach is straightforward in comparison to the traditional VM-SP approach. The present method directly solves various real option problems by naturally reformulating the optimality conditions as a GLCP system, whereas the traditional approach derives the VM-SP conditions held at each free-boundary by imposing certain restrictive assumptions specified for each problem, and solutions are indirectly derived by solving the VM-SP conditions. The VM-SP approach is therefore applicable *only* to these simplified (or restricted) real option problems, whereas the present approach can be used to solve more complicated and practical problems as well as these simplified problems.

Second, the GLCP approach presented in this paper can be applied to other types of real option problems involving more complicated situations, for example, a time-to-build option with capacity choice (Bar-Ilan et al., 2002), a repeated real option (Malchow-Møller and Thorsen, 2005); or a multi-option (e.g. Trigeorgis, 1993).

Third, although the mathematics and algorithm presented in this article may, at first glance, appear esoteric and thus inaccessible to readers who do not have a specialized interest in advanced mathematical theory, this approach may in fact be broadly applied to solve a range of complicated, practical real option problems in economics and finance. For example, our approach may be implemented in an integrated modeling system such as GAMS (General Algebraic Modeling System), which is specifically designed for modeling linear, nonlinear and complementarity problems and widely used to analyze economic models, e.g. computational general equilibrium models, macro economic models, etc. Such an implementation will not require enormous exertion because the present framework and algorithm here is not only efficient, but is also systematic.

Finally, the analysis in this article can be exploited as a building-block for studying a novel class of real option problems, termed as "option graphs," which can be interpreted as a generalized version of real option with flexibility, as described by Kulatilaka (1995) for example. The option graph is a compound real option consisting of decision making whose interdependent structure is represented as a general directed graph. We refer the interested readers to our companion paper, Akamatsu and Nagae (forthcoming). This companion paper provides a general algorithm, in which the present algorithm [Algo-GLCP] is used as a subprocedure.

Appendix A A Review: Numerical Methods for Pricing Options with Timing Choice

This appendix reviews the existing numerical methods for pricing option problems with timing choice, i.e., American options. The first rigorous mathematical formulation of the American option pricing problem is given by McKean (1965), Bensoussan (1984) and Karatzas (1988), where an American option problem is formulated as an *optimal stopping problem*, or equivalently, a *free boundary problem*. Since closed-form solutions are not available for these problems, an extensive literature of numerical methods to approximate the option price and exercise strategies has been developed by discretizing time and the state space as shown in Section 4.

In spite of the vast collection of numerical recipes that exist to solve American option problems [3], none of these methods seems to be applicable for options with complicated exercise structures, such as [P-A] and [P-B], in general settings. The reason is that the above existing methods have at least one of the following two serious disadvantages: First, some of the earlier (but still widely used) methods may be inappropriate even for the plain vanilla American option in terms of their accuracy and efficiency. Second, the other methods are based on mathematical structures specific to the plain vanilla American options, and so are not capable of treating more complicated real options such as [P-A]. In what follows, we briefly describe the existing methods for solving the American option problems and outline their disadvantages.

The existing methods are roughly classified into the following three categories: a) explicit methods; b) LCP methods; and c) other miscellaneous approaches.

Explicit methods are among the earliest methods used to price both financial and real options, and remain among the most widely used. In these methods, the option price —– approximated in a discretized framework as shown in Section 4 —– at the ith time-step is calculated by simple substitutions of the option prices at the $(i+1)$th time-step: for the case of a plain vanilla American put option with strike price K, the option value at ith time-step at each state is calculated by the following procedure:

$$V^{i,j} := \max.\left\{\phi^j(\boldsymbol{V}^{i+1}), \quad P^j - K\right\}, \quad \forall j = 1, \cdots, \mathrm{J}. \tag{A.1}$$

where $\phi^j(\boldsymbol{V}^{i+1}) \triangleq \mathbb{E}\left[e^{-\rho\Delta t}V^{i+1}\middle|P(t^i) = P^j\right]$ and $\boldsymbol{V}^i \triangleq \{V^{i,1}, \cdots, V^{i,\mathrm{J}}\}$ for any i. The expectation of $\phi^j(\cdot)$ is calculated by using either the binomial approximation of the underlying process, or the finite-difference approximation of the partial differential equations resulting from the Feynman-Kac formula. For more details, we refer the reader to Cox et al. (1979), Boyle (1988), Hull (1989), and Duffie (1996). The explicit method characterized by (A.1) can be regarded as a straightforward extension of the Euler explicit scheme for solving partial differential equations to the system of Bellman equations for the American put:

$$V(t,P) = \max.\{\phi(t,P;V), \quad P - K\}, \quad \forall(t,P) \in [0,T] \times \mathcal{R}_{++}, \tag{A.2}$$

where $\phi(t,P;V) \triangleq \lim_{\Delta\downarrow 0} \mathbb{E}\left[e^{-\rho\Delta}\{V(t,P) + \Delta V(t,P)\}\middle|P(t) = P\right]$ and $\Delta V(t,P)$ is an increment of the value function during Δ. The explicit method thus inherits the disadvantages of the explicit Euler finite-difference scheme, which make the explicit method inappropriate even for the plain vanilla problem in terms of its poor discrete-to-continuous convergence: the approximate solutions obtained from the discretized model may not converge to their continuous counterpart unless certain conditions

[3] The (plain vanilla) American option here is defined as a single option which can be exercised at any time; The option is killed and never restored when the exercise is carried out, unlike the cyclic option [P-A], where the option to enter the market is restored whenever the firm leaves the market.

are satisfied (Amin and Khana, 1994; Lamberton, 1998; Leisen, 1998; and Jaillet et al., 1990). Unfortunately, satisfying the conditions required to guarantee the discrete-to-continuous convergence can make the computational procedure very demanding, as described by Huang and Pang (1998), Dempster and Hutton (1999), and Coleman et al. (2002).

In contrast to the explicit method, the *linear complementarity problem* (LCP) method, which can be regarded as a special case of the present method, achieves both accuracy and efficiency under reasonably mild conditions. In the LCP method, the system of Bellman equations (A.2) of a plain vanilla American is formulated as a system of LCPs, or equivalent *variational inequality problems*. For the theoretical foundation, we refer to Jaillet et al. (1990), Myneni (1992) and Dempster and Hutton (1999). Numerical methods for the LCP approach are developed by Brennan and Schwartz (1977), Wilmott, Dewynne, and Howison (1993), Huang and Pang (1998), and Coleman et al. (2002). Several studies have demonstrated that the LCP approach is more efficient than the explicit methods for the case of a plain vanilla American option with the underlying state variable following a geometric Brownian motion (Huang and Pang, 1989; Dempster and Hutton, 1999; and Coleman et al., 2002). Despite its remarkable advantages, the LCP approach is not directly applicable to [P-A] and [P-B], whose optimality conditions reduce to a system of GLCPs rather than standard LCPs.

There are several other numerical methods for pricing vanilla American options, such as the quasi-analytical solution method (Geske and Johnson, 1984; Barone-Adesi and Whaley, 1987; and MacMillan, 1986) and the integral representation method (Kim, 1990; Jacka, 1991; Broadie and Detemple, 1996; and Detemple and Tian, 2002). Since these methods are based on the specific mathematical structure associated with vanilla options with a single free boundary, it appears to be difficult to directly apply these methods to the cyclic option models such as [P-A], where multiple free boundaries exist.

It is worthwhile to note that there are several naïve expansions of the explicit method to more complex real option problems such as the cyclic option [P-A] (e.g. Trigeorgis, 1991 and Kulatilaka and Trigeorgis, 1994). Unfortunately, these methods *not only* inherit the above disadvantages of the explicit method *but also* cause an inconsistency: for the case of the cyclic option [P-A], the (expanded) explicit method computes the value function at ith time-step, $(V_0^{i,j}, V_1^{i,j})$, by the following procedure.

$$\begin{cases} V_0^{i,j} := \max. \left\{ \phi_0^i(\boldsymbol{V}_0^{i+1}), \quad \phi_1^i(\boldsymbol{V}_1^{i+1}) - C_E \right\}, \\ V_1^{i,j} := \max. \left\{ \phi_1^i(\boldsymbol{V}_1^{i+1}), \quad \phi_0^i(\boldsymbol{V}_0^{i+1}) - C_Q \right\}, \end{cases} \qquad j = 1, \cdots, \mathrm{J}, \tag{A.3}$$

where $\phi_m^i(\boldsymbol{V}_m^i) \triangleq \mathbb{E}\left[e^{-\rho\Delta t} V_m^{i+1} \middle| P(t^i) = P^j, m(t^i) = m \right]$ and $\boldsymbol{V}_m^i \triangleq \{V_m^{i,1}, \cdots, V_m^{i,\mathrm{J}}\}$ for $m = 0, 1$. It is clear that procedure (A.3) is inconsistent with the Bellman equation of [P-A] at each moment of time:

$$\begin{cases} V_0(t,P) = \max.\{\phi_0(t,P;V), \quad V_1(t,P) - C_E\}, \\ V_1(t,P) = \max.\{\phi_1(t,P;V), \quad V_0(t,P) - C_Q\}, \end{cases} \quad \forall P \in \mathcal{R}_{++}, \tag{A.4}$$

where $\phi_m(t,P;V) \triangleq \lim_{\Delta\downarrow 0} \mathbb{E}\left[e^{-\rho\Delta}\{V_m(t,P) + \Delta V_m(t,P)\}\middle| P(t) = P, m(t) = m\right]$ for $m = 0,1$. It should be emphasized that the solution of the Bellman equation system (A.4), $V_0(t,P)$ and $V_1(t,P)$, should be determined *simultaneously* whatever the finite-difference scheme is. According to this fact, the present method is quite natural and perhaps the most simple for solving the cyclic option [P-A]: at each time-step, it directly solves the Bellman equation (A.4) as a GLCP, and obtains the value functions $\boldsymbol{V}_0^i$ and $\boldsymbol{V}_1^i$ simultaneously. To the best of our knowledge, thus far there have been no other numerical methods developed to solve the Bellman equation for the cyclic option (A.4) without requiring inconsistent approximations like (A.3).

Appendix B Algorithm for [GLCP]

This appendix shows the algorithm of Peng and Lin (1999) for solving the subproblem [GLCP]. They employ a truncated Newton method in order to accelerate its local convergence. Specifically, the inner loop of the outer iteration k, for solving the system of smooth equations, $\boldsymbol{H}(\boldsymbol{V}, \xi^{(k)}) = \boldsymbol{0}$, is truncated after a single iteration. The details of their algorithm are as follows:

[Algo-Peng-Lin]

Step 0. Given constant numbers $\epsilon_0 \geq 0$, $\omega \in (0,1)$, $\delta_1 \in (0,1)$, $\delta_2 \in (0,1)$, $\delta_3 \in (0, 1-\delta_2)$.
Choose any $\xi^{(0)} > 0$, $\boldsymbol{V}^{(0)} \in \mathcal{R}^{\mathrm{J}}$ and $\gamma \geq \frac{\|\boldsymbol{H}(\boldsymbol{V}^{(0)},\xi^{(0)})\|}{\min\{\xi^{(0)},1\}}$.

Step 1. The Newton step of $\boldsymbol{H}(\boldsymbol{V}, \xi^{(k)})$:
If $\nabla_{\boldsymbol{V}}\boldsymbol{H}(\boldsymbol{V}^{(k)}, \xi^{(k)})$ is singular, STOP. (the algorithm fails);
else if $\|\boldsymbol{H}(\boldsymbol{V}^{(k)})\| \leq \epsilon_0$, STOP. ($\boldsymbol{V}^{(k)}$ is an appropriate solution of [GLCP]);
Otherwise, compute a Newton step $\boldsymbol{d}^{(k)}$ satisfying Eq. (37).

Step 2. Compute $\boldsymbol{V}^{(k+1)}$:
Let $h^{(k)}$ be the maximum value of $\{1, \delta_1, \delta_1^2, \cdots\}$ such that

$$\|\boldsymbol{H}(\boldsymbol{V}^{(k)} + h^{(k)}\boldsymbol{d}^{(k)}, \xi^{(k)})\| \leq (1 - \omega h^{(k)}) \min\{\xi^{(k)}, 1\}\gamma, \tag{B.1}$$

and $\boldsymbol{V}^{(k+1)} := \boldsymbol{V}^{(k)} + h^{(k)}\boldsymbol{d}^{(k)}$.

Step 3. Compute $\xi^{(k+1)}$:
If $\left(\boldsymbol{V}^{(k+1)}, \min.\{\delta_3, \xi^{(k)}\}\xi^{(k)}\right) \in \mathcal{N}\left(\gamma, \min.\{\delta_3, \xi^{(k)}\}\xi^{(k)}\right)$ then set $\upsilon^{(k)} := 1 - \min.\{\delta_3, \xi^{(k)}\}$
;

Otherwise, let $\upsilon^{(k)}$ be the maximum value of $\{\delta_2, \delta_2^2, \cdots\}$ such that

$$\left(\boldsymbol{V}^{(k+1)}, (1-\upsilon^{(k)})\xi^{(k)}\right) \in \mathcal{N}\left(\gamma, (1-\upsilon^{(k)})\xi^{(k)}\right). \tag{B.2}$$

Set $\xi^{(k+1)} := (1-\upsilon^{(k)})\xi^{(k)}$.
Step 4. $k := k+1$, return to Step 1.

where in the above

$$\mathcal{N}(\beta, \xi) = \left\{(\boldsymbol{V}, \xi) \in \mathcal{R}^{\mathrm{J}} \times \mathcal{R}_+ \middle| \|\boldsymbol{H}(\boldsymbol{V}, \xi)\| \leq \beta \min.\{\xi, 1\}\right\}. \tag{B.3}$$

Appendix C Standard LCP Representation of A Plain Vanilla American Option Problem

This appendix shows that for the case that the real option problem [P-A] is a plain vanilla American option, this problem reduces to a system of *standard* LCPs. Consider a problem with the same configuration as [P-A] in Section 2, but that there is no option to leave the market: a firm that is in the market can not exit from the market. In this case, the value of a firm that is in the market $\{\boldsymbol{V}_1^i\}$ can be regarded as a given constant: the unknown variable is given by the value of a firm that is outside of the market (or, equivalently, the value of an option to enter the market), $\{\boldsymbol{V}_0^i\}$. The option value at the ith time-step can be obtained as the solution to the following GLCP given $\boldsymbol{V}_0^{i+1}$ and $\boldsymbol{V}_1^i$.
[GLCP-A'(D)] Find $\boldsymbol{V}_0$ such that

$$\begin{cases} (-\boldsymbol{L}\boldsymbol{V}_0 - \bar{\boldsymbol{g}}) \cdot \left(\boldsymbol{V}_0 - \bar{\boldsymbol{V}}_1 + \mathbf{1}_{\mathrm{J}} C_E\right) = 0, \\ -\boldsymbol{L}\boldsymbol{V}_0 - \bar{\boldsymbol{g}} \geq \mathbf{0}, \quad \boldsymbol{V}_0 - \bar{\boldsymbol{V}}_1 + \mathbf{1}_{\mathrm{J}} C_E \geq \mathbf{0}, \end{cases}$$

where the suffix i is omitted for notational simplicity. Here $\bar{\boldsymbol{g}} \triangleq \boldsymbol{M}^i \boldsymbol{V}_0^{i+1}$ and $\bar{\boldsymbol{V}}_1 \triangleq \boldsymbol{V}_1^i$ are regarded as given vectors. By introducing a new variable $\boldsymbol{Y} \triangleq \boldsymbol{V}_0 - \bar{\boldsymbol{V}}_1 + \mathbf{1}_{\mathrm{J}} C_E$, we can reduce [GLCP-A'(D)] to a standard LCP:

[LCP-A'] Find $\boldsymbol{Y}$ such that $\boldsymbol{Y} \cdot \boldsymbol{F}(\boldsymbol{Y}) = 0, \quad \boldsymbol{Y} \geq \mathbf{0}, \quad \boldsymbol{F}(\boldsymbol{Y}) \geq \mathbf{0},$

where $\boldsymbol{F}(\boldsymbol{Y}) \triangleq -\boldsymbol{L}(\boldsymbol{Y} + \bar{\boldsymbol{V}}_1 - \mathbf{1}_{\mathrm{J}} C_E) - \bar{\boldsymbol{g}}$.

However, it may be noted that the above variable transformation does not work in the case of cyclic option, since both $\boldsymbol{V}_0$ and $\boldsymbol{V}_1$ are unknown variables.

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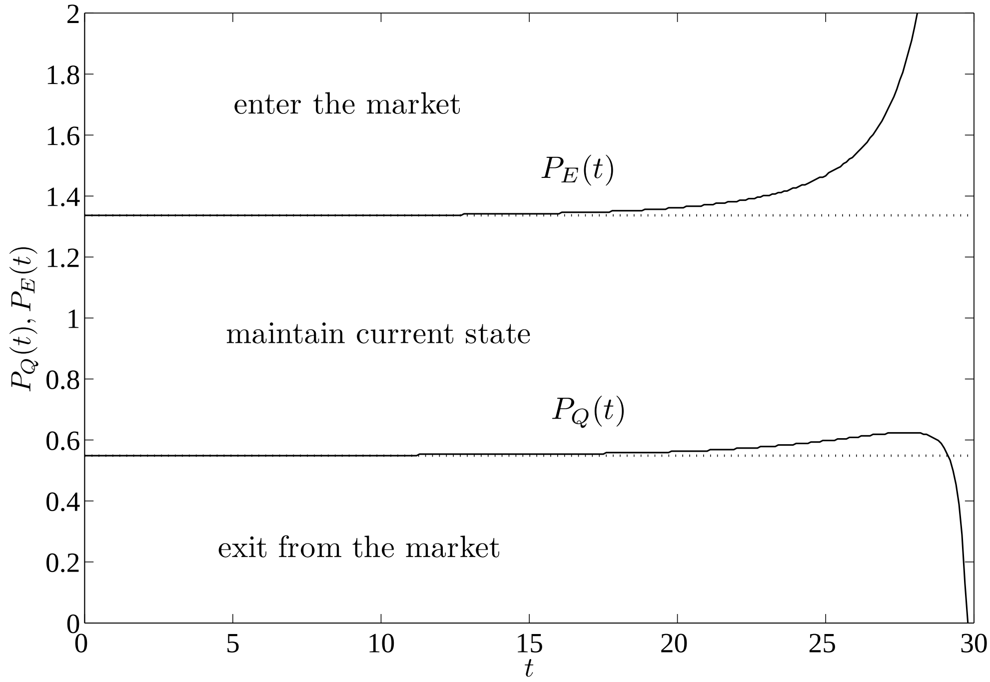

Fig. 1. Entry and exit thresholds as functions of t. At any moment $t \in [0, T]$, a firm that is outside the market enters the market when the market price of the product $P(t)$ exceeds a certain threshold $P_E(t)$, whereas a firm that is in the market leaves the market when the price falls below the other threshold $P_Q(t)$. When $P(t) \in [P_Q(t), P_E(t)]$, the firm neither enters nor leaves the market. The two dotted lines indicate these thresholds obtained in the infinite horizon framework of Dixit and Pindyck (1994, Chap.7).

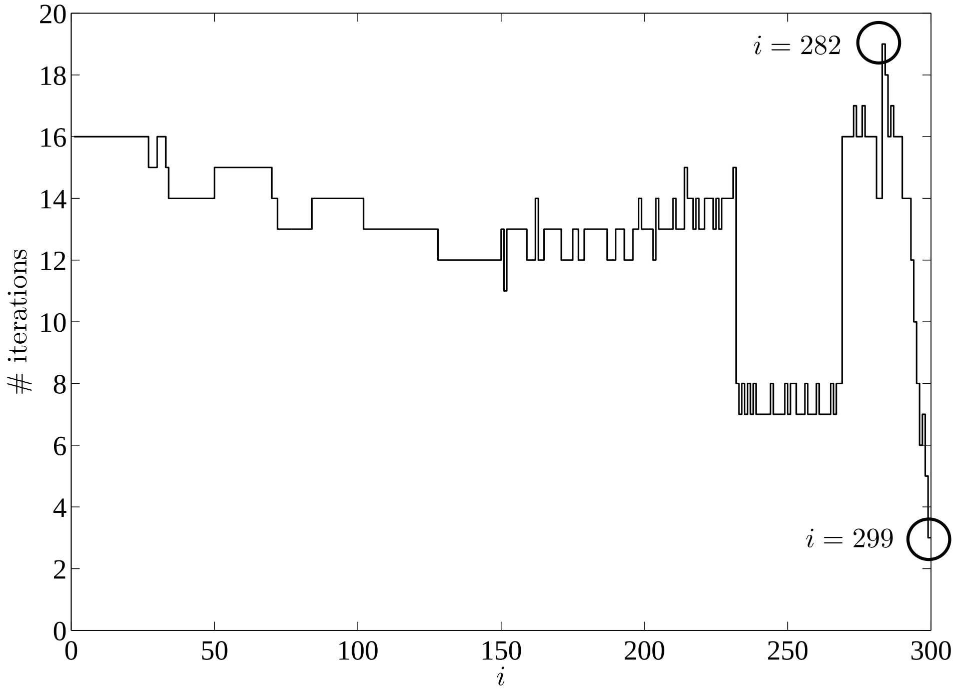

Fig. 2. Number of iterations required to solve the subproblems of [GLCP-A(D)]. The horizontal axis represents the index of subproblems, *i*, and the vertical axis represents the number of iterations required for solving the subproblem. The 299th subproblem requires the smallest number of iterations to be solved, whereas the 282nd subproblem requires the largest number of iterations.

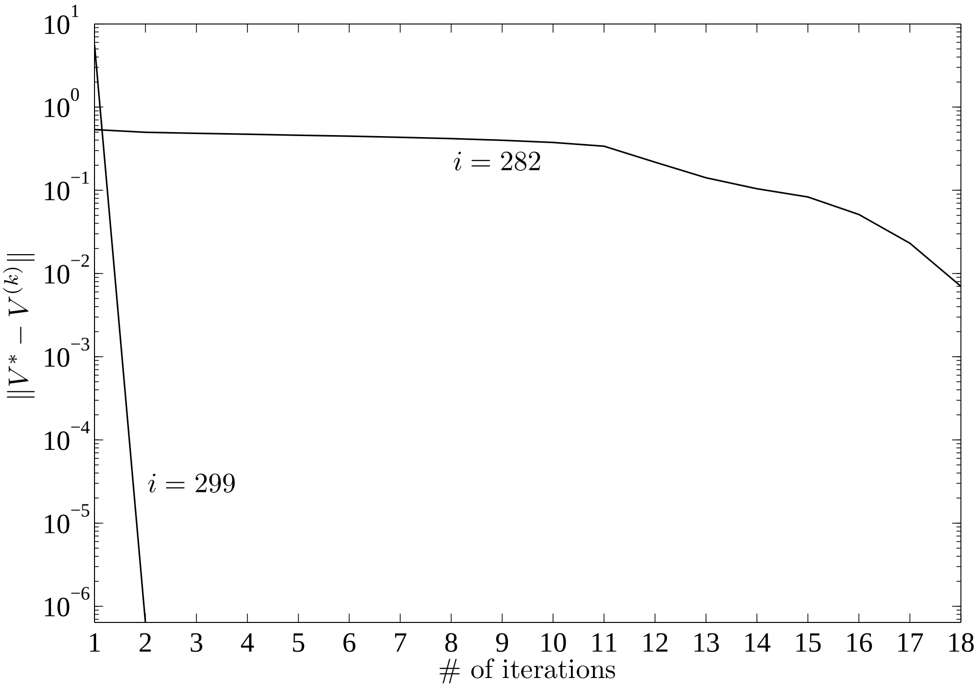

Fig. 3. Convergence patterns of the Peng-Lin algorithm for the subproblems [GLCP-A^{299}] and [GLCP-A^{282}]. The horizontal axis represents the number of iterations, k, and the vertical axis represents the logarithm of the difference between the temporal solution $\boldsymbol{V}^{(k)}$ and the exact solution of the subproblem, $\boldsymbol{V}^{*}$.

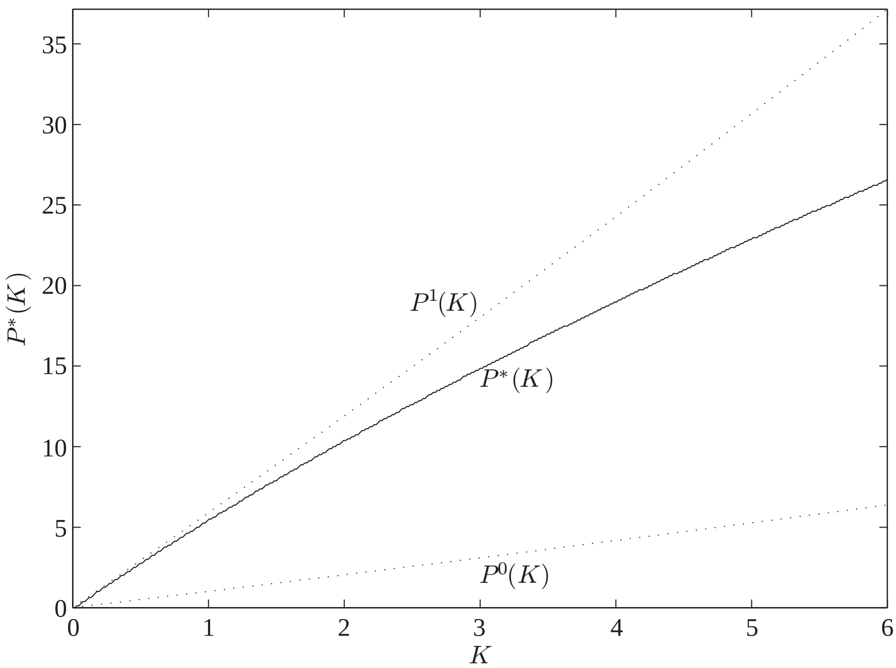

Fig. 4. Investment thresholds as functions of the remaining expenditure K. $P^{*}(K)$ is the commencement-suspension threshold of the time-to-build option [P-B]. $P^{(0)}(K)$ is the investment threshold for the naïve NPV-rule, while $P^{1}(K)$ is the commencement threshold for a building project that cannot be suspended once commenced.

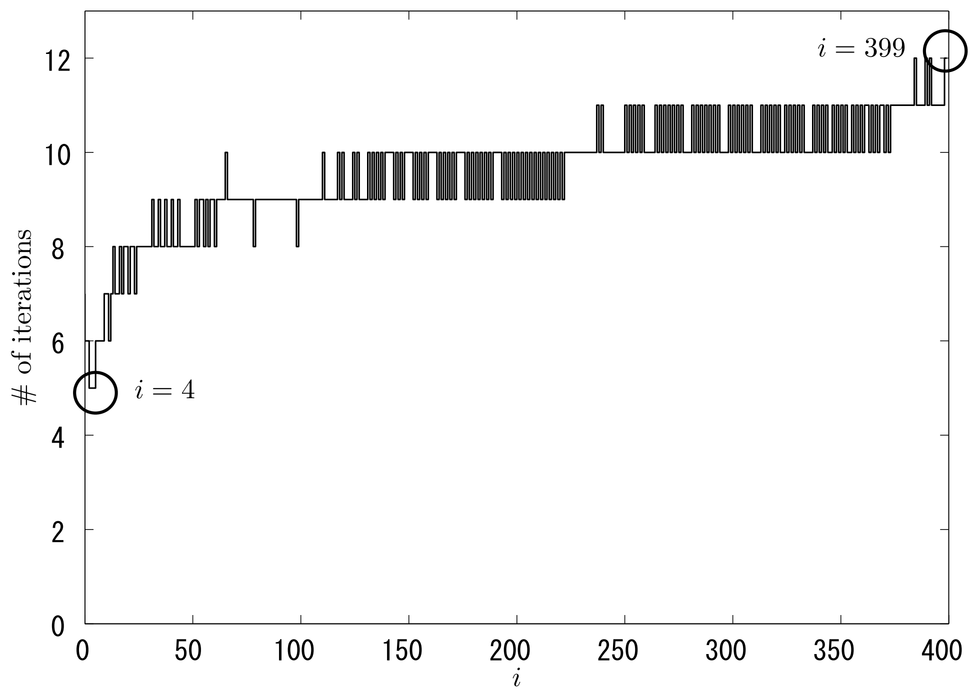

Fig. 5. Number of iterations required to solve the subproblems of [GLCP-B(D)]. The horizontal axis represents the index of subproblems, *i*, and the vertical axis represents the number of iterations required for solving the subproblem. The 4th subproblem requires the smallest number of iterations to be solved, whereas the 399th subproblem is solved in the largest number of iterations.

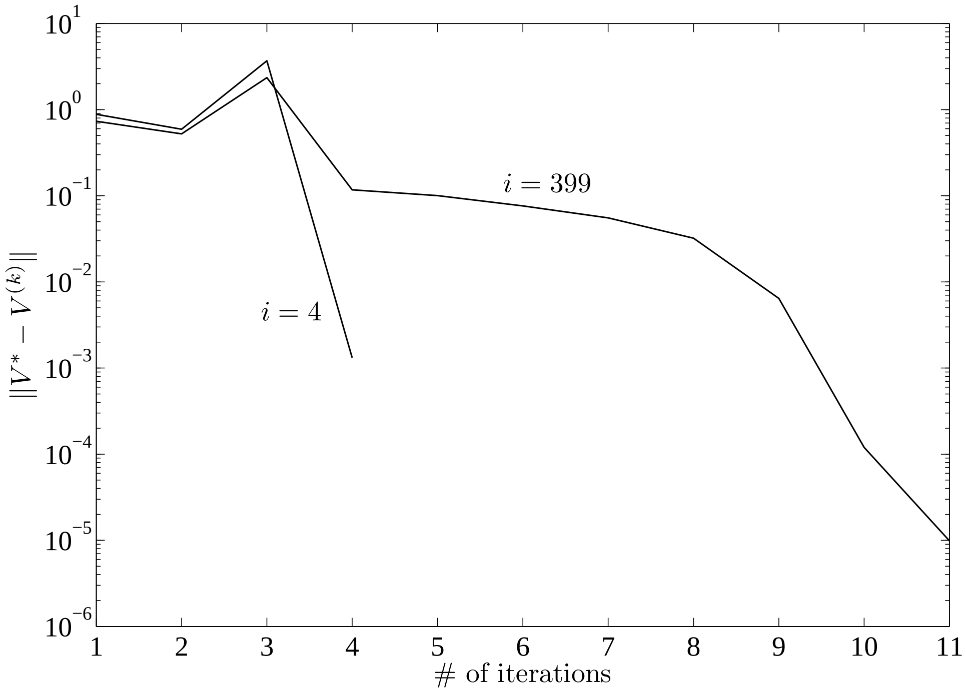

Fig. 6. Convergence patterns of the Peng-Lin algorithm for the subproblems [GLCP-B^4] and [GLCP-B^{399}]. The horizontal axis represents the number of iterations, k, and the vertical axis represents the logarithm of the difference between the smoothing path $\boldsymbol{V}^{(k)}$ and the exact solution of the subproblem, $\boldsymbol{V}^*$.