DYNAMIC REVENUE MANAGEMENT OF A TOLL ROAD PROJECT UNDER TRANSPORTATION DEMAND UNCERTAINTY

Takeshi Nagae
Graduate School of Science and Technology, Kobe University
1-1 Rokko-dai, Nada, Kobe, Hyogo 657-8501, JAPAN
nagae@kobe-u.ac.jp
Tel: +81-78-803-6448 Fax: +81-78-803-6360

Takashi Akamatsu
Graduate School of Information Sciences, Tohoku University
Aoba, Aoba-ku, Sendai, Miyagi 980-8579, JAPAN
Phone: +81-22-795-7503 Fax: +81-22-795-7505

Abstract

This study proposes a prototype quantitative method for dynamic revenue management of a private toll road, taking into account the long-term dynamics of transportation demand. This is first formulated as a stochastic singular control problem, in which the manager can choose the toll level from two discrete values. Each toll change requires nonnegative adjustment costs. Our analysis then reveals that the optimality condition reduces to standard linear complementarity problems, by using certain function transformation techniques. This enables us to develop an efficient algorithm for solving the problem, exploiting the recent advances in the theory of complementarity problems.

Keywords

Toll road projects, dynamic revenue management, transportation demand uncertainty, stochastic singular control, generalized complementarity problem

Recently, private toll roads have received a fair amount of attention as an alternative to public-free-access roads. This private provision appears to be motivated by shortages of public funds, particularly in developing countries, as well as the fact (or belief) that the private sector is likely to be more efficient than the public sector. The main concern of the private sector would be the maximization of the ENPV (*expected net present value*) of the cash flow streams—normally comprising the revenue from the road toll charge, maintenance costs and operation cost—within the operating period. In general, the revenue from the toll road fluctuates due to the long-term dynamics of the transportation demand of the toll road over the period. From such a viewpoint, in the examination of the profitability of a project with a long-operating duration, the day-to-day demand dynamics and feedback intertemporal decision-making should be more dominant as compared to within-day dynamics. Despite this significance, neither the dynamic uncertainty of transportation demand nor intertemporal decision-making has received satisfactory treatment in the road transportation literature.

The discussion regarding private toll roads in the literature of transportation could be categorized into three directions. The first direction pertains to expansion or generalization of the marginal pricing theory after Dupuit and was developed by Beckman (1965), Dafermos and Sparrow (1971) and Smith (1979). Yang and Huang (1998) and Yang (1999) investigated the marginal cost pricing in a general network. Yang and Huang (1997) analyzed a time-varying toll model of a road bottleneck using optimal control theory. The second direction of this discussion involves Yang and Huang's (1997) examination of efficiency from the viewpoint of social welfare and their analysis of both profitability and efficiency among various ownership regimes of the private toll road. De Palma and Lindsey (2000, 2002) examined the profitability of time-based congestion tolling on a bottleneck using Vickrey's queuing congestion model (Vickrey (1969)) and compared a broad variety of private ownership regimes, including the Nash and Stackelberg equilibria for a mixed duopoly. Yang and Meng (2000) examined both private sector profit and the total social welfare gain under either monopolistic (with a single private operator) or perfect competitive (among many private operators) markets of toll road services in a static framework. The third and final direction concerns the relationship between the self-financing of capacity investment and toll road revenue. Arnott and Kraus (1995) discussed the finance of capacity investments in the context of Vickrey's bottleneck model and investigated the circumstances under which the time-varying pricing should be self-financing. Yang and Meng (2000) formulated a quantitative model as a network design problem in which the road toll and road capacity are jointly optimized. All the abovementioned studies consider either static frameworks or within-day dynamic frameworks with bottleneck congestion.

The ENPV maximization from a private toll road project could be identified as a revenue management problem. In the field of revenue management, the daily (monthly, quarterly, yearly, or a considerably longer unit of time) dynamics of demand and its uncertainty play a prominent part, and the stochastic control approach has also been widely used (see e.g. Talluri and van Ryzin (2004)). However, nearly all the relevant studies that employ the stochastic control theory concentrate on a situation in which each control variable can take any value within a certain range at each moment of time. This continuity assumption could sometimes be unrealistic, particularly in the case where positive costs are required for adjusting the control variables. For instance, it is natural to assume that the private toll road manager can change the toll among a set of several discrete levels, and switching the toll each time entails a certain cost (e.g. advertisement costs to notify the users of the toll change). This kind of problem, in which the control variables can take only discrete values, is referred to as a class of *singular* stochastic control problem first introduced by Karatzas (1983). It is evident that the standard approach of optimal control with continuous control variables is no longer applicable to a singular control problem, and thus, the characteristics of the singular control problem are fairly different from those of the standard control problem. Interested readers can refer to Kushner and Martins (1991), Kushner and Dupuis (1992) and Kumar and Muthuraman (2004). The stochastic singular control approach thus assumes importance and has a wide variety of potential applications; nevertheless, it has attracted little attention in the fields of both revenue management and transportation science.

This paper proposes a prototype framework for analyzing the dynamic revenue management problems of a toll road, taking into account the long-term dynamics of the transportation demand. In our framework, the manager is assumed to intermittently switch the toll between a given set of discrete levels, depending on the transportation demand that is continuously observed at each moment of time. We formulate a stochastic singular control problem and show that its optimality condition can be rewritten as a sequence of GCP (*generalized complementarity problem*). Our analysis reveals that certain function transformation reduces the GCP to a standard LCP (*linear complementarity problem*). This enables us to develop an efficient algorithm for solving the problem in a successive manner, exploiting the recent advances in linear complementarity theory. These are our original contributions not only in the field of transportation science but also in those of revenue management and stochastic optimization.

The remainder of this paper is organized as follows. Section 1 formulates the dynamic revenue management of a toll road as a stochastic singular control problem. Section 2 denotes the optimality condition as a sequence of GCPs. Section 3 shows the reduction of each GCP to a standard LCP in an appropriate discrete time-state framework. A numerical method is developed by using this result. Section 4 shows several numerical examples, and Section 5 provides some concluding remarks.

1 Model

Suppose a toll road project and a manager who operates the road for a certain operating period, $[0,T]$. The road will be transferred to the public sector at the end of the operation[1]. At each moment of time[2], the manager is assumed to observe the transportation demand (number of vehicles per unit time), $q(t)$, and chooses the toll charge level, $c(t)$, as either c_L (the lower toll) or c_H (the higher toll). We denote the set of tolls as $C \equiv \{c_L, c_H\}$ and assume $c_L < c_H$. We further assume that a fixed adjustment cost $I_{L,H}$ ($I_{L,H}$) is incurred when the manager switches the toll level from c_L to c_H (c_H to c_L).

The transportation demand $q(t)$ is assumed to vary stochastically over time following a stochastic differential equation (SDE). In order to describe our basic concept intuitively, we begin with a discrete-time framework. Let $0 = t^0, t^1, \cdots, t^i, \cdots, t^I = T$ be a discrete time grid with interval Δt (e.g. a day), and let $q_0, q_1, \cdots, q_i, \cdots, q_I$ denote the demand at each point of time. Suppose that we observe the demand $q_i = \bar{q}$ and choose the toll level as $c_t = \bar{c}$ at the i th date (here, the variables with bar represent a certain or observed value.) At that time, the demand at the $i+1$ th date, q_{i+1}, is uncertain; hence, the increment of the demand $\Delta q_i \equiv q_{i+1} - q_i = q_{i+1} - \bar{q}$ is a random variable. It is natural to assume that the increment Δq_i has a mean and variance, each of which is proportional to length of the interval, Δt. We assume the increment to be

$$\Delta q_i = \alpha(q_i, c_i)\Delta t + \sigma(q_i, c_i)\Delta W_i \quad , \tag{1}$$

where c_i is the toll charge level chosen at the i th date. The first term on the right-hand side of (1) is the deterministic part of the increment and the second represents the random part, which captures the demand uncertainty. ΔW_i is assumed to be a normal random variable with mean 0 and variance Δt. Observe that $E[\Delta q_i] = \alpha(q_i, c_i)\Delta t$ and $\mathrm{Var}[\Delta q_i] = \{\sigma(q_i, c_i)\}^2 \Delta t$.

In equation (1), $\alpha : \mathrm{R}_+ \times C \to \mathrm{R}$ is a known function, which might represent the trend of the transportation demand, the seasonal cyclic pattern, the relation between the transportation demand and the travel cost of the toll road and so on. For example, let us show how the travel time and its effect on the demand are involved. First, we assume that the generalized travel cost of the toll road comprises the travel time $\tau(q_i)$—an increasing function of the demand—and the toll charge level c_i at the i th date. It is natural to assume that the transportation demand will increase (decrease) when the current travel cost $\tau(\bar{q}) + \bar{c}$ is sufficiently low (high). This can be implemented by letting $\alpha(q,c)$ be a decreasing function with respect to $\tau(q)+c$, that is, $\alpha(q,c) \equiv f[\tau(q)+c]$. On the other hand, in equation (1), $\sigma : \mathrm{R}_+ \times C \to \mathrm{R}_+$ is also assumed to be a known function, which represents the degree of uncertainty of the transportation demand; $\sigma \to 0$ implies that we can completely predict the future demand, and a large σ implies that the demand fluctuates widely. Our framework is sufficiently generalized and the specifications of $\alpha(q,c)$ and $\sigma(q,c)$, which might be inevitable for the application of our method to the actual problems, is not essential (see Section 4 for an example of such a specification and Section 5 for further discussion).

For the purpose of mathematical tractability, we consider the abovementioned dynamics in a continuous time framework. We first rewrite equation (1) as a stochastic differential equation:

$$\mathrm{d}\,q(t) = \alpha[q(t), c(t)]\mathrm{d}t + \sigma[q(t), c(t)]\mathrm{d}W(t), \quad q(0) = q_0 = \text{given const.}, \tag{2}$$

where $\alpha : \mathrm{R}_+ \times C \to \mathrm{R}$ and $\sigma : \mathrm{R}_+ \times C \to \mathrm{R}_+$ are given functions. It is naturally assumed that $\alpha(q, c_L) \geq \alpha(q, c_H)$ $\forall q \in \mathrm{R}_+$, since a lower toll charge encourages transportation demand. We further assume that $\alpha(0,c)=\sigma(0,c)=0,\ \ \forall c\in C$ in order to preclude negative transportation demand. In equation (2), $\mathrm{d}W(t)$ is an increment of a standard Brownian motion defined on an appropriate probability space $(\Omega,\mathcal{F},\mathcal{P})$, where $\Omega,\mathcal{F}$ represent the sample space and its σ-algebra, respectively, and $\mathcal{P}$ is the probability measure[3].

Note that the manager chooses the toll charge level $c(t)$ at time t corresponding to the observed demand $q(t)$ and the current toll level $c(t)$[4], both of which are not revealed until t. This implies that the toll strategy takes the form of a function $c:[0,T]\times \mathrm{R}_+\times C\rightarrow C$. In the following, we denote a toll strategy by $c(\cdot)\equiv\{c(t,q,c)\,|\,(t,q,c)\in[0,T]\times\mathrm{R}_+\times C\}$.

The manager intends to maximize the ENPV of cash flow streams during the operating period $[0,T]$, by choosing the toll strategy $c(\cdot)$. This is formulated as the following stochastic singular control problem:

$$\text{[P]}\qquad \max_{c(\cdot)} E\left[J(0,T,c(\cdot),\omega)\,\middle|\,(0,q_0,c_0)\right],$$

where $E[\,\cdot\,|\,(t,q,c)]$ is an expectation conditional to the information set available at t, $(q(t),c(t))=(q,c)$, and c_0 is the initial toll level. $J(t,T,c(\cdot),\omega)$—the net present value of cash flow streams during $[t,T]$ under toll strategy $c(\cdot)$ with respect to sample path $\omega\in\Omega$—is defined as

$$J(t,T,c(\cdot),\omega)\equiv\int_t^T e^{-\rho(s-t)}\left\{\pi[q(s),c(s)]-\sum_{n,m} I_{n,m}\delta_{n,m}(s)\right\}\mathrm{d}s\,, \tag{3}$$

where ρ is the discount rate. $\pi[q(t),c(t)]\equiv c(t)q(t)$ is the instantaneous revenue[5] per unit time at time t. $\delta_{n,m}(t)$ is a delta function, which takes $1/\mathrm{d}s$ if the toll is switched from c_n to c_m at t, and 0, otherwise.

2 Optimality condition

This section derives the optimality condition of the dynamic revenue management problem [P] by using the DP (*dynamic programming*) principle. In our framework, the optimality condition is represented as a system of GCPs. First, we define the value function of the problem [P] when the transportation demand $q(t)=q$ is observed and toll level $c(t)=c$ is chosen at time t, as follows:

$$V(t,q,c) \equiv \max_{c(\cdot)} E\left[J(t,T,c(\cdot),\omega) \middle| (t,q,c)\right], \qquad \forall (t,q,c) \in [0,T] \times \mathrm{R}_{+} \times C, \tag{4}$$

where $J(t,T,c(\cdot),\omega)$ is the net present value (evaluated at t) of cash flow streams in the remaining duration $[t,T]$ defined in equation (3). Note that the value function $V:[0,T]\times \mathrm{R}_{+} \times C \to \mathrm{R}_{+}$ itself as well as the optimal strategy $c^{*}:[0,T]\times R_{+} \times C \to C$ are unknown functions.

In order to derive the optimality condition, let us suppose that the transportation demand $q(t)=q$ is observed and the higher toll $c(t)=c_H$ is chosen at time t. By applying the DP principle, we observe that the manager takes one of the following two actions: either switches the toll from the current to another, thereby incurring the adjustment cost or defers it for a certain time. When the manager does not change the toll level for a sufficiently small time Δ, it must be true that

$$V(t,q,c_H) \geq \pi(q,c_H)\Delta + e^{-\rho\Delta} \max_{c(\cdot)} E\left[\, J(t+\Delta,T,c(\cdot),\omega) \mid (t,q,c_H)\,\right]. \tag{5}$$

Taking $\Delta \to 0$ and using Ito's lemma, we have the following partial differential inequality.

$$-\pi(q,c_H) - \mathrm{L}_H V(t,q,c_H) \geq 0. \tag{6}$$

See Appendix A for more a detailed derivation. In equation (6), L_n—an infinitesimal generator of the transportation dynamics described in (2) when the toll level is c_n— is defined as a partial differential operator as follows:

$$\mathsf{L}_n V(t,q,c_n) \equiv \left\{ \frac{\partial}{\partial t} + \alpha(q,c_n) \frac{\partial}{\partial q} + \frac{1}{2} \{\sigma(q,c_n)\}^2 \frac{\partial^2}{\partial q^2} - \rho \right\} V_n(t,q,c_n) .$$

On the other hand, if the manager, chooses to switch the toll level from c_H to c_L, the value function should satisfy

$$V(t,q,c_H) \geq V(t,q,c_L) - I_{H,L} \tag{7}$$

or

$$V(t,q,c_H) - V(t,q,c_L) + I_{H,L} \geq 0 . \tag{8}$$

Since one of the two actions must be optimal, either equation (8) or (6) hold as equal. Hence, the optimality at the state (t,q,c_H) is

$$\begin{cases} -\mathsf{L}_H V(t,q,c_H) - \pi(q,c_H) > 0 \text{ and } V(t,q,c_H) - V(t,q,c_L) + I_{H,L} = 0, \\ -\mathsf{L}_H V(t,q,c_H) - \pi(q,c_H) = 0 \text{ and } V(t,q,c_H) - V(t,q,c_L) + I_{H,L} > 0, \end{cases}$$

or

$$\min\left\{ -\mathsf{L}_H V(t,q,c_H) - \pi(q,c_H),\ V(t,q,c_H) - V(t,q,c_L) + I_{H,L} \right\} = 0 . \tag{9}$$

Similarly, let us consider the case where the lower toll level—$c(t) = c_L$ instead of c_H—is chosen. In this case too, the manager can take one of two actions, i.e. either switch to the toll level from c_L to c_H, or do nothing during Δ. When the manager chooses to maintain the current toll level, it must be true that

$$-\mathsf{L}_L V(t,q,c_L) - \pi(q,c_L) \geq 0 .$$

Meanwhile, if the manager changes the toll level from the current to another, it must be true that

$V(t,q,c_L) - V(t,q,c_H) + I_{L,H} \geq 0$.

Hence, the optimality condition at the state (t,q,c_L) can be represented by

$$\min\left\{-\mathcal{L}_L V(t,q,c_L) - \pi(q,c_L),\ V(t,q,c_L) - V(t,q,c_H) + I_{L,H}\right\} = 0 . \tag{10}$$

It is noteworthy that, theoretically, the manager can change $c(t)$ any number of times. This implies that the value function for the higher toll level, $V(t,q,c_H)$, cannot be determined without using the value function for the lower toll level, $V(t,q,c_L)$, and vice versa, at any state $(t,q) \in [0,T] \times \mathbb{R}_+$. Therefore, conditions (9) and (10) should hold simultaneously, and the value function V should be obtained as a solution of the system of GCPs:

[GCP-∞] Find $V \equiv \left\{V(t,q,c) \mid (t,q,c) \in [0,T] \times \mathbb{R}_+ \times C\right\}$ such that

$$\begin{cases} \min\left\{-\mathcal{L}_H V(t,q,c_H) - \pi(q,c_H),\ V(t,q,c_H) - V(t,q,c_L) + I_{H,L}\right\} = 0 \\ \min\left\{-\mathcal{L}_L V(t,q,c_L) - \pi(q,c_L),\ V(t,q,c_L) - V(t,q,c_H) + I_{L,H}\right\} = 0 \end{cases}, \quad \forall (t,q) \in [0,T) \times \mathbb{R}_+ .$$

Since the manager is assumed to transfer the road to the government at the end of operation, the terminal condition held at T is denoted as

$$V[T,q(T),c(T)] = 0, \qquad \forall [q(T),c(T)] \in \mathbb{R}_+ \times C . \tag{11}$$

The optimal strategy eventually takes the following intuitively plausible form: the toll is switched to c_L (or not switched if it is already $c(t) = c_L$) when the demand $q(t)$ falls below the threshold, $q_L^*(t)$, and is switched to c_H (or maintained at $c(t) = c_H$) when the demand $q(t)$ exceeds another threshold, $q_H^*(t)$ at time t. The toll should not be changed as long as the demand remains $q_L^*(t) \leq q(t) \leq q_H^*(t)$, regardless of the toll level.

3 Reduction to a Linear Complementarity Problem

(1) Discretization

Since [GCP ∞] cannot be solved analytically, the solution of the dynamic revenue management problem [P] should be obtained numerically. Therefore, we first reformulate [GCP ∞] in a discrete framework. Suppose a sufficiently large subspace $[q_{\min}, q_{\max}]$ in the state (transportation demand) space R_+. We then consider a discrete grid in the time-state space $[q_{\min}, q_{\max}] \times [0, T]$ with increments Δt and Δq. Let $(t^i, q^j) \equiv (i\Delta t, j\Delta q + q_{\min})$ be each point of the grid, where the indices $i = 0,1,\cdots I$ and $j = 0,1,\cdots, J, J+1$ characterize the locations of the point with respect to time and state, respectively. We also denote the value function $V(t, q, c_n)$ and the instantaneous profit $\pi(q, c_n)$ at a grid point (t^i, q^j) by $V_n^{i,j}$ and π_n^j, respectively.

In this framework, L_n can be approximated by using an appropriate finite-difference scheme (e.g. that of Crank-Nicholson), as follows:

$$\mathsf{L}_n V(t^i, q, c_n) \approx \mathbf{L}_n \mathbf{V}_n^i + \mathbf{M}_n \mathbf{V}_n^{i+1},$$

where $\mathbf{V}_n^i \equiv [V_n^{i,1} \ \cdots V_n^{i,J}]^{\mathrm{T}}$ is a J-dimensional column vector with the jth element the value function corresponding to the toll c_n and the demand q^j at time t^i, and $\mathbf{L}_n$ and $\mathbf{M}_n$ are $J \times J$ square matrices determined by the transportation demand process (2). See Appendix B for detailed definitions of $\mathbf{L}_n$ and $\mathbf{M}_n$. Then, each subproblem of [GCP ∞] held at time t^i can be given as

[GCP ^{i}] Find $\mathbf{V}^i \equiv \{\mathbf{V}_H^i, \mathbf{V}_L^i\} \in \mathrm{R}^{2J}$ such that

$$\begin{cases} \min\left\{-\mathbf{L}_H \mathbf{V}_H^i - \mathbf{M}_H \mathbf{V}_H^i - \boldsymbol{\pi}_H,\ \ \mathbf{V}_H^i - \mathbf{V}_L^i + \mathbf{1}^J I_{H,L}\ \right\} = \mathbf{0}^J \\ \min\left\{-\mathbf{L}_L \mathbf{V}_L^i - \mathbf{M}_L \mathbf{V}_L^i - \boldsymbol{\pi}_L,\ \ \mathbf{V}_L^i - \mathbf{V}_H^i + \mathbf{1}^J I_{L,H}\ \right\} = \mathbf{0}^J \end{cases},$$

where $\mathbf{0}^J$ and $\mathbf{1}^J$ denote nth column vectors with all elements equal to 0 and 1, respectively. Similarly, we can rewrite the terminal condition at time $t^I = T$ given by equation (11) as

$$\mathbf{V}_H^I = \mathbf{V}_L^I = \mathbf{0}^J. \tag{12}$$

Note that the subproblem [GCP^i] is independent from other subproblems [GCP^j] $i \neq j$ when $\mathbf{V}^{i+1}$ is known. This characteristic reveals that the series of subproblems $\{[\text{GCP}^i] \mid i = 0,1,\cdots I\}$ can be solved in a successive manner as follows: i) using the terminal condition $\mathbf{V}_H^I = \mathbf{V}_L^I = \mathbf{0}^J$, solve the subproblem [GCP^{I-1}] and obtain the solution $\mathbf{V}^{I-1}$; ii) using $\mathbf{V}^{I-1}$ as a given constant, solve the subproblem [GCP^{I-2}] and obtain $\mathbf{V}^{I-2}$; and iii) repeating the procedure recursively, obtain the entire value function $\{\mathbf{V}^i \mid i = 0,1,\cdots, I\}$. Thus, we should focus on the methods to solve each [GCP^i] separately, rather than simultaneously.

(2) Reduction to a Standard Linear Complementarity Problem

The subproblem [GCP^i] is still difficult to solve, even numerically, because the problem is not in standard form. Therefore, this section shows the reduction of [GCP^i] to a standard LCP by using certain variable transformation techniques.

Supposing that the value functions $\mathbf{V}_H^{i+1}$ and $\mathbf{V}_L^{i+1}$ are known when we solve [GLP^i], consider the following variable transformation.

$$\mathbf{X}_n^i \equiv -\mathbf{L}_n \mathbf{V}_n^i - \mathbf{g}_n^i, \qquad \forall n \in \{H, L\},\ \ \forall i \in \{0,1,\cdots I-1\}, \tag{13}$$

where $\mathbf{g}_n^i \equiv \mathbf{M}_n \mathbf{V}_n^{i+1} + \boldsymbol{\pi}_n$ is a given constant. Assuming that the matrix $\mathbf{L}_n$ is nondegenerated, we can represent the value function $\mathbf{V}_n^i$ as a linear function of $\mathbf{X}_n^i$, that is,

$$\mathbf{V}_n^i = \mathbf{V}_n^i\left(\mathbf{X}_n^i\right) \equiv -\mathbf{L}_n^{-1}\mathbf{X}_n^i - \mathbf{h}_n^i, \ \forall n \in \{H, L\}, \ \forall i \in \{0,1,\cdots,I-1\}, \tag{14}$$

where $\mathbf{h}_n^i \equiv -\mathbf{L}_n^{-1}\mathbf{g}_n^i$ is a given constant.

Substituting equations (13) and (14) into [GCP ^{i}], we obtain the following standard LCP, where the unknown variables are only $\mathbf{X}_H^i$ and $\mathbf{X}_L^i$.

[LCP ^{i}] Find $\mathbf{X}^i$ such that

$$\mathbf{X}^i \cdot \mathbf{H}^i\left(\mathbf{X}^i\right) \equiv 0, \qquad \mathbf{X}^i \geq \mathbf{0}^{2J}, \ \mathbf{H}^i\left(\mathbf{X}^i\right) \geq \mathbf{0}^{2J},$$

where $\mathbf{X}^i \equiv \begin{bmatrix} \mathbf{X}_H^i \\ \mathbf{X}_L^i \end{bmatrix}$, $\mathbf{H}^i\left(\mathbf{X}^i\right) \equiv \begin{bmatrix} -\mathbf{L}_H^{-1} & \mathbf{L}_L^{-1} \\ \mathbf{L}_H^{-1} & -\mathbf{L}_L^{-1} \end{bmatrix}\begin{bmatrix} \mathbf{X}_H^i \\ \mathbf{X}_L^i \end{bmatrix} + \begin{bmatrix} -\mathbf{h}_H^i + \mathbf{h}_L^i + \mathbf{1}I_{H,L} \\ -\mathbf{h}_L^i + \mathbf{h}_H^i + \mathbf{1}I_{L,H} \end{bmatrix}$.

Since [LCP ^{i}] is in standard form, we can develop an efficient algorithm for solving the problem by exploiting the recent advances in the linear complementarity theory (see Ferris and Pang (1997)). Due to space constraints, we have omitted the proofs of existence and uniqueness of [LCP ^{i}]. Interested readers can refer to Nagae and Akamatsu (2004).

If $\mathbf{X}^i$ —the solution of [LCP ^{i}] $\mathbf{X}^i$ —is obtained, we can easily calculate the original unknown variable (i.e. the solution of the subproblem [GLP ^{i}])— $\mathbf{V}^i$ — via reverse variable transformation, as shown in equation (14). We can now summarize the algorithm for solving dynamic revenue management [P] as follows:

Step 0 Set $\mathbf{V}_H^I = \mathbf{V}_L^I = \mathbf{0}$ and $i := I-1$.

Step 1 If $i < 0$, then STOP.

Step 2 Obtain $\mathbf{X}^i$ as the solution of [LCP ^{i}] by regarding $\mathbf{V}^{i+1}$ as a given constant.

Step 3 Calculate $\mathbf{V}^i$ via reverse variable transformation (equation (14)).

Step 4 Set $i := i-1$ and return to **Step 1**.

4 Numerical Solution

In this section, we report the numerical results. This section aims to provide an illustrative example of our framework and to examine whether the aforementioned algorithm functions appropriately. First, we specify the transportation demand process $q(t)$ as a mean-reverting process:

$$\mathrm{d}\,q(t) = \mu\left\{\overline{D}_n - q(t)\right\}\mathrm{d}t + \sigma q(t)\,\mathrm{d}W(t), \quad n = H, L. \quad q(0) = q_0 = \text{given.} \tag{15}$$

The first term on the right-hand side of the equation indicates that the transportation demand reverts to the values $\overline{D}_H$ and $\overline{D}_L$ that correspond to the toll levels c_H and c_L, respectively. In this case, $\overline{D}_H$ ($\overline{D}_L$) can be recognized as the long-term mean or the steady-state of the transportation demand when the manager set and maintained the toll level at c_H (c_L). In equation (15), μ is a given constant which represents the convergence speed of the transportation demands to the long-term means $\overline{D}_H$ and $\overline{D}_L$. The second term on the right-hand side of (15) represents the random part of the demand σ, precluding any possibilities of negative demand. A constant σ implies the size of randomness.

The parameters are set as $T = 20$, $\mu = 0.2$, $\sigma = 20\%$, $\rho = 10\%$, $\overline{D}_H = 0$, $\overline{D}_L = 1$, $c_H = 1.5$, $c_L = 1$, $I_{L,H} = 1$ and $I_{H,L} = 0.2$, and the dynamic revenue management problem [P] is then solved.

Figure 1 plots the value functions $V_L(t,q)$ and $V_H(t,q)$, for each $c(t) \in \{c_L, c_H\}$ at the initial time $t = 0$, as functions of the initial transportation demand q_0. The thresholds

$q_L^*(t)$ and $q_H^*(t)$, which yield an optimal toll strategy as discussed in Section 2, at the initial time $t = 0$ are also illustrated in Figure 1.

The evolution of $q_L^*(\mathrm{t})$ and $q_H^*(t)$ is shown in Figure 2. Observe that both the thresholds decrease with time because as time passes, the manager prefers to yield instantaneous profits by choosing the higher toll, rather than increase the transportation demand by choosing the lower toll. This is due to a time lag between the toll changes and their effect on transportation demand.

Figure 1

Figure 2

5 Conclusion

We proposed a prototype framework for the quantitative analysis of dynamic revenue management of a toll road project involving transportation demand risk. We first formulated a dynamic revenue management problem as a stochastic singular control problem, the optimality condition for which is written as a set of GCPs. We then revealed that each GCP reduces to a standard LCP via certain variable transformation techniques in an appropriate discrete framework. Subsequently, we developed an efficient algorithm by exploiting recent advances in linear complementarity theory. Finally, several numerical examples were illustrated.

It should be noted that this study is designed as an initial step towards developing a useful quantitative framework for managing private toll roads with dynamic uncertainty of the transportation demand. Our framework can be expanded and generalized into a more complex and realistic model (e.g. a model in which the network user equilibrium is achieved at each moment of time) without any severe difficulties and major modifications of its mathematical structure. Future work can focus on analyzing such an extended model.

Another important and interesting topic would pertain to the social welfare of the private toll road project as discussed in De Palma and Lindsey (2000, 2002). The present methodology provides us with a useful basis for examining the social welfare gain of the dynamic revenue management of private toll roads.

Appendix A

When the manager chooses to maintain the current toll level for Δ, it must be true that

$$\begin{aligned}
V(t,q,c_n) &\geq \max_{c(\cdot)} E\Big[\int_t^{t+\Delta} e^{-\rho(s-t)}\pi(q(s),c_n)ds \\
&\qquad + \int_{t+\Delta}^{T} e^{-\rho(s-t)}\Big\{\pi(q(s),c(s)) + \sum_n \sum_m \delta_{n,m}(s) I_{n,m}\Big\}ds \;\Big|\; (t,q,c_n)\Big] \\
&= E\Big[\int_t^{t+\Delta} e^{-\rho(s-t)}\pi(q(s),c_n)ds \\
&\qquad + \max_{c(\cdot)} E\Big[\int_{t+\Delta}^{T} e^{-\rho(s-t)}\Big\{\pi(q(s),c(s)) + \sum_n \sum_m \delta_{n,m}(s) I_{n,m}\Big\}ds \;\Big|\; (t,q,c_n)\Big]\Big|(t,q,c_n)\Big] \\
&= E\Big[\int_t^{t+\Delta} e^{-\rho(s-t)}\pi(q(s),c_n)ds + e^{-\rho\Delta} \max_{c(\cdot)} E\big[J(t+\Delta,T,c(\cdot),\omega)\,\big|(t,q,c_n)\big]\Big|(t,q,c_n)\Big],
\end{aligned}$$

where we used the following nested structure of conditional expectation for the second arrangement:

$$E\big[\cdot\,\big|(t,q,c_n)\big] = E\big[E\big[\cdot\,\big|(t,q,c_n)\big]\big|(t,q,c_n)\big].$$

When Δ is sufficiently small, the last equation can be rewritten as

$$V(t,q,c_n) \geq \pi(q,c_n)\Delta + \frac{1}{1+\rho\,\Delta} E\big[V[t+\Delta, q+\Delta q, c_n]\,|\,(t,q,c_n)\,\big],$$

where Δq is the increment of the demand during Δ. We denote the last term on the right-hand side of equation (5) as $E\big[V[t+\Delta, q+\Delta q, c_n]\,|\,(t,q,c_n)\big] \equiv V(t,q,c_n) + E[\Delta V\,|\,(t,q,c_n)]$, where ΔV is the increment of the value function during Δ. A simple rearrangement yields

$$\rho\, V(t,q,c_n) \geq (1+\rho\Delta)\pi(q,c_n) + \frac{E[\Delta V\,|\,(t,q,c_n)]}{\Delta}.$$

Taking $\Delta \to 0$, we have

$$\rho\, V(t,q,c_n) \ge \pi(q,c_n) + \lim_{\Delta \to 0} \frac{E[\Delta V \mid (t,q,c_n)]}{\Delta} \tag{16}$$

According to Ito's lemma, the last term on the right-hand side of (16) can be rewritten as

$$\lim_{\Delta \to 0} \frac{E[\Delta V | (t,q,c_n)]}{\Delta} = \frac{\partial V(t,q,c_n)}{\partial t} + \alpha(q,c_n)\frac{\partial V(t,q,c_n)}{\partial q} + \frac{1}{2}\{\sigma(q,c_n)\}^2 \frac{\partial^2 V(t,q,c_n)}{\partial q^2}. \tag{17}$$

Substituting (17) into (16), we obtain equation (6).

Appendix B

We first approximate each partial differential in [GCP∞] by the Crank-Nicholson scheme in the following discrete time-state framework:

$$\frac{\partial V(t^i,q^j,c_n)}{\partial t} \approx \frac{V_n^{i+1,j} - V_n^{i,j}}{\Delta t},\quad \frac{\partial V(t^i,q^j,c_n)}{\partial q} \approx \frac{1}{2}\left\{\frac{V_n^{i+1,j+1} - V_n^{i+1,j-1}}{2\Delta q} + \frac{V_n^{i,j+1} - V_n^{i,j-1}}{2\Delta q}\right\},$$

$$\frac{\partial^2 V(t^i,q^j,c_n)}{\partial q^2} \approx \frac{1}{2}\left\{\frac{V_n^{i+1,j+1} - 2V_n^{i+1,j} + V_n^{i+1,j-1}}{(\Delta q)^2} + \frac{V_n^{i,j+1} - 2V_n^{i,j} + V_n^{i,j-1}}{(\Delta q)^2}\right\}.$$

Substituting these approximations in the definition of L_n, we have

$$L_n V(t^i,q,c_n) \approx \mathbf{L}_n \mathbf{V}_n^i + \mathbf{M}_n \mathbf{V}_n^{i+1},$$

where

$$\mathbf{L}_n \equiv \begin{bmatrix} b^1 & c^1 & 0 & & 0 & 0 & 0 \\ a^2 & b^2 & c^2 & \cdots & 0 & 0 & 0 \\ 0 & a^3 & b^3 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & b^{J-2} & c^{J-2} & 0 \\ 0 & 0 & 0 & \cdots & a^{J-1} & b^{J-1} & c^{J-1} \\ 0 & 0 & 0 & & 0 & a^J & b^J \end{bmatrix},\quad \mathbf{M}_n \equiv \begin{bmatrix} d^1 & c^1 & 0 & & 0 & 0 & 0 \\ a^2 & d^2 & c^2 & \cdots & 0 & 0 & 0 \\ 0 & a^3 & d^3 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & d^{J-2} & c^{J-2} & 0 \\ 0 & 0 & 0 & \cdots & a^{J-1} & d^{J-1} & c^{J-1} \\ 0 & 0 & 0 & & 0 & a^J & d^J \end{bmatrix},$$

and $a^j \equiv \frac{1}{2}\left(-\frac{\alpha_n^j}{2\Delta q}+\frac{1}{2}\left(\frac{\sigma_n^j}{\Delta q}\right)^2\right)$, $b^j \equiv -\rho-\frac{1}{\Delta t}-\frac{1}{2}\left(\frac{\sigma_n^j}{\Delta q}\right)^2$, $c^j \equiv \frac{1}{2}\left(\frac{\alpha_n^j}{2\Delta q}+\frac{1}{2}\left(\frac{\sigma_n^j}{\Delta q}\right)^2\right)$ and

$d^j \equiv \frac{1}{\Delta t}-\frac{1}{2}\left(\frac{\sigma_n^j}{\Delta q}\right)^2$.

Notes

[1] This assumption is not essential. Certain terminal payoffs (that might be a function of the demand at the end of the operation, say, $\Pi(q(T))$) can be involved in our framework without any difficulties.

[2] We use the terms "time" and "moment" in the standard stochastic control theory even though the length of unit time (e.g. day, week or month) appears rather long to be referred to as moment.

[3] For the fundamental terminology of probability theory, refer to the standard textbooks of either finance or stochastic process theory, for example, Duffie (1992) and Øksendal (1998).

[4] More precisely, it refers to the toll charge level *just before* time t, i.e. $\lim_{\delta \to 0} c(t-\delta)$ instead of the current toll charge level, $c(t)$.

[5] We omit the operating cost (e.g. maintenance costs) at this point for notation simplification. Our framework can be easily extended to a case with such an operating cost.

References

Arnott, Richard and Marvin Kraus. (1995). "Financing Capacity in the Bottleneck Model." Journal of Urban Economics 38, 272–290.

Beckmann, M. J. (1965). "On Optimal Tolls for Highways, Tunnels and Bridges." In L. Edie, R. Herman, and R. Rothery (eds.), Vehicular Traffic Science: American Elsevier, pp. 331–341.

Dafermos, S.C. and F.T. Sparrow. (1971). "Optimal Resource Allocation and Toll Patterns in User-optimized Transportation Network." Journal of Transportation Economics and Policy 5, 198–200.

de Palma, Andre and Robin Lindsey. (2000). "Private Toll Roads: Competition Under Various Ownership Regimes." Annals of Regional Science 34, 13–35.

de Palma, Andre and Robin Lindsey. (2002). "Private Roads, Competition, and Incentives to Adopt Time-based Congestion Tolling." Journal of Urban Economics 52, 217–241.

Duffie, Darrell. (1992). Dynamic Asset Pricing Theory. Princeton: Princeton University Press.

Ferris, Michael C. and Jong-Shi Pang. (1997). "Engineering and Economic Applications of Complementarity Problems." SIAM Review 39, 669–713.

Fukushima, Masao. (1992). "Equivalent Differentiable Optimization Problem and Descent Methods for Asymmetric Variational Inequality Problems." Mathematical Programming 53, 99–110.

Karatzas, Ioannis. (1983). "A Class of Singular Stochastic Control Problems." Advances in Applied Probability 15, 225–254.

Kumar, Sunil and Kumar Muthuraman. (2004). "A Numerical Method for Solving Singular Stochastic Control Problems." Operations Research, Vol. 52, No. 4, pp. 563–582.

Kushner, Harold J. and Luiz Felipe Martins. (1991). "Numerical Methods for Stochastic Singular Control Problems." SIAM Journal on Control and Optimization 29, 1443–1475.

Kushner, Harold J. and Dupuis, Paul G. (1992). Numerical Methods for Stochastic Control Problems in Continuous Time. New York: Springer.

Nagae, Takeshi and Takashi Akamatsu. (2004). "A Stochastic Control Model of Infrastructure Project Decisions Represented as a Graph Structure." Journal of Infrastructure Planning and Management 772(IV-65), 185–202 (in Japanese).

Øksendal, Bernt. (1998). Stochastic Differential Equations: An Introduction with Applications: Springer-Verlag.

Smith, M.J. (1979). "The Marginal Cost Taxation of a Transportation Network." Transportation Research B 13, 237–242.

Talluri, Kalyan T. and Garrett J. Van Ryzin. (2004). The Theory and Practice of Revenue Management: Kluwer Academic Publishers.

Vickrey,William S. (1969). "Congestion Theory and Transport Investment." American Economic Review 59, 251–260.

Yang, Hai and Hai-Jun Huang. (1997). "Analysis of the Time-varying Pricing of a Bottleneck with Elastic Demand using Optimal Control Theory." Transportation Research B 31, 425–440.

Yang, Hai and Qiang Meng. (2000). "Highway Pricing and Capacity Choice in A Road Network under a Build-operate-transfer Scheme." Transportation Research A 34, 207–222.

Yang, Hai. (1999). "System Optimum, Stochastic User Equilibrium, and Optimal Link Tolls." Transportation Science 33, 354–360.

Figures

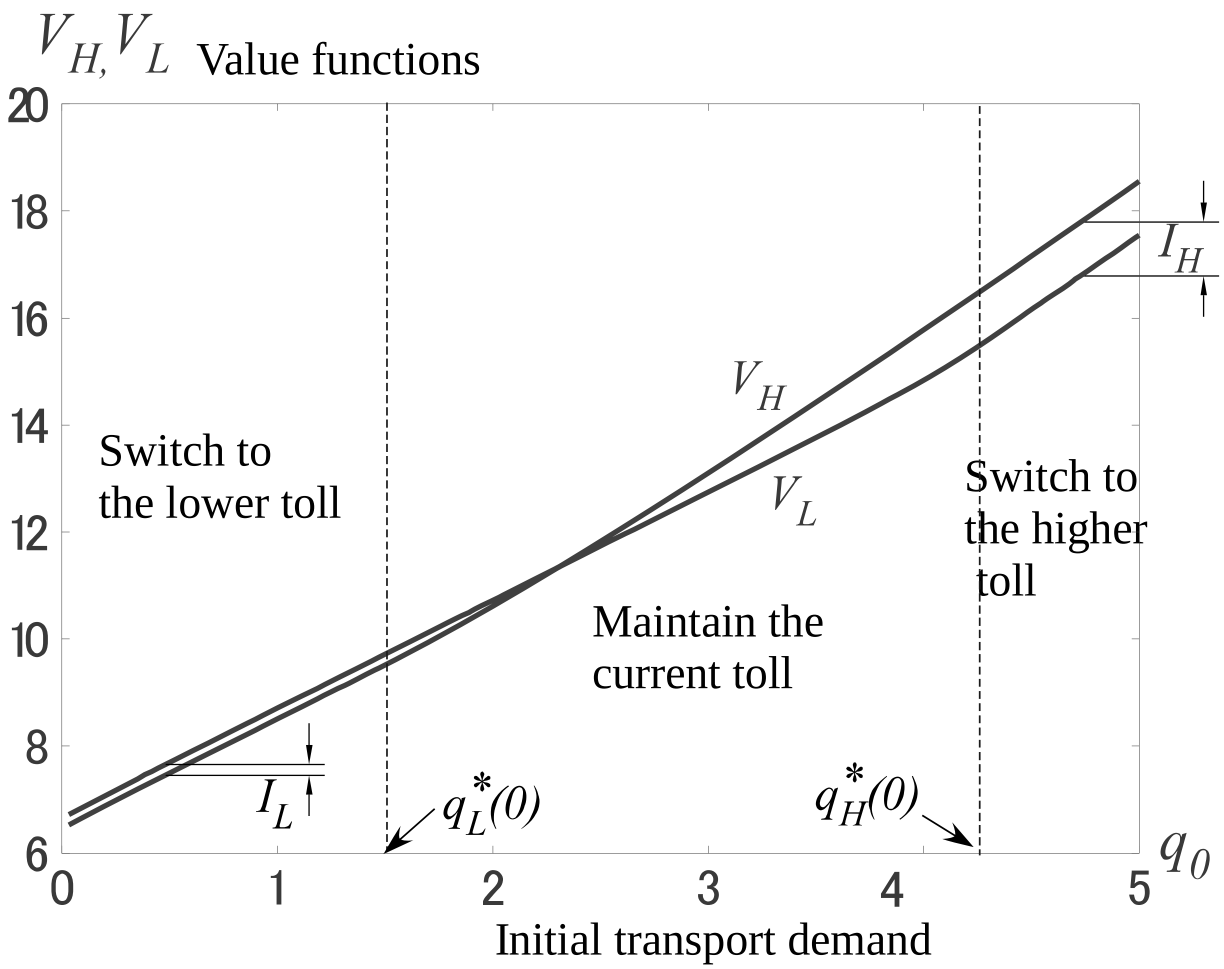

Figure 1. Value functions

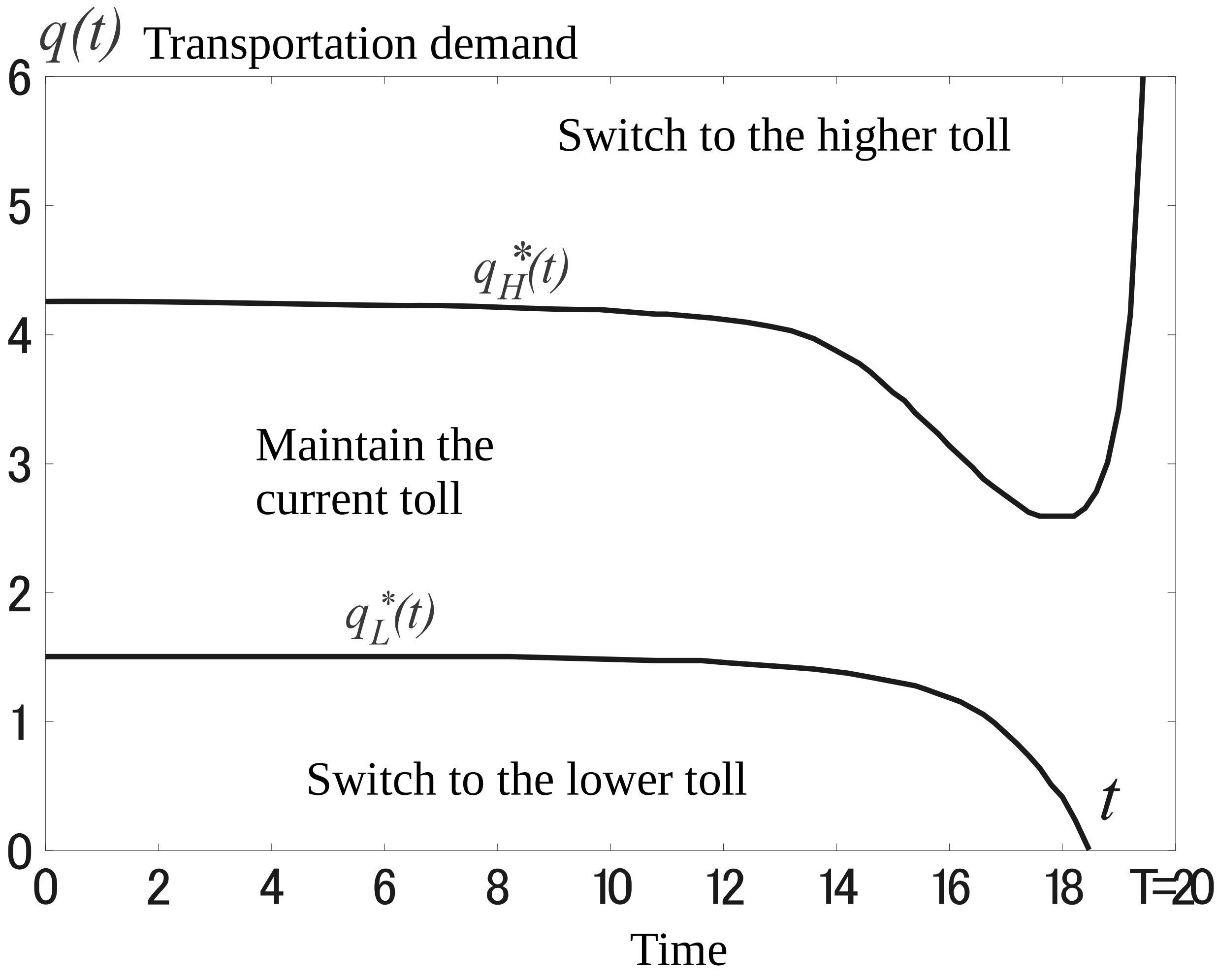

Figure 2. Optimal toll switching strategy