Detecting Dynamic Traffic Assignment
Capacity Paradoxes in Saturated Networks

Takashi AKAMATSU

Graduate School of Information Sciences, Tohoku University
Aoba, Sendai, Miyagi 980-8579, JAPAN (E-mail: akamatsu@plan.civil.tohoku.ac.jp)

Benjamin HEYDECKER

Centre for Transport Studies, University College London
Gower Street, London, WC1E 6BT, ENGLAND   (E-mail: ben@transport.ucl.ac.uk)

Abstract

Creation of a new link or increase in capacity of an existing link can reduce the efficiency of a congested network as measured by the total travel cost. This phenomenon, of which an extreme example is given by Braess's paradox, has been examined in conventional studies within the framework of static assignment. For dynamic traffic assignment, which makes account of the effect of congestion through explicit representation of queues, Akamatsu (2000) gave a simple example of the occurrence of this paradox. The present paper extends that result to a more general network. We first present a necessary and sufficient condition for the paradox to occur in a general network in which there is a queue on each link. We then give a graph-theoretic interpretation of the condition, which gives us a convenient method to test whether or not the paradox will occur by performing certain tests on information that describes the network structure. Finally, as an application of this theory, we examine several example networks and queueing patterns where occurrence of this paradox is inevitable.

Creation of a new link or increase in capacity of an existing link in a congested network can worsen the efficiency and increase the total cost of travel. This phenomenon, which is known as "Braess' paradox" (Braess, 1968) or "Smith's paradox"(Smith,1978), has been studied intensively under static equilibrium traffic assignment framework in conventional studies (for example, Murchland (1970), Fisk (1979), Frank (1981), Steinberg and Zangwill (1983), Dafermos and Nagurney (1984), Harker (1988), Catoni and Pallottino (1991), Alperovich (1993), Pas and Principio (1997), Yang and Bell (1998)).

However, many matters concerning the paradox in non-stationary settings are yet to be investigated because the properties of dynamic flows with queues are significantly different from those of static flows. Recently, some attempts have been made to study the possibility of the paradox under the dynamic traffic flows with queues: Arnott *et al* (1993) showed that users' departure time choice behaviour also can cause the paradox; Daganzo (1998) conceptually discussed the paradox as well as a "hyper sensitivity" problem arising from combination of queue spillover phenomena and users' route choice behaviour; Akamatsu (2000) gave an explicit example for the paradox under dynamic user equilibrium (DUE) traffic assignment, which is a natural generalisation of the well-known static user equilibrium assignment.

The purpose of this paper is to explore the general properties of the capacity paradox shown in Akamatsu (2000). More specifically, we first establish a necessary and sufficient condition for the paradox to occur in "saturated networks" with general structure, in which there is a queue on each link. We then give a graph theoretic interpretation of the condition, which gives us a convenient method to test whether or not the paradox occurs from certain information on network structure. Finally, as an application of this theory, we examine several typical networks where the paradox occurrence is inevitable.

The organisation of the paper is as follows. In the next section, we briefly review the DUE assignment, restricting ourselves to the minimum knowledge required for considering our problem. In Section **2,** we establish a necessary and sufficient condition for occurrence of the paradox in saturated networks. In Section **3**, we identify a graph theoretic interpretation of the condition, and finally, concluding remarks are made in Section **4**.

1. PRELIMINARIES - Dynamic User Equilibrium Assignment

1.1 Networks and Notation

Our model is defined on a transportation network $G[N, L, W]$ consisting of the set L of directed links with L elements, the set N of nodes with N elements, and the set W of origin-destination (OD) nodes pairs. The origins and destinations are subsets of N, and we denote them by R and S, respectively. An origin/destination node with no arriving/leaving link is called "pure origin/destination". Throughout this paper, *we consider only networks with a one-to-many OD* pattern (*ie R* has a single element). Sequential integers from 1 to N are allocated to the nodes in N. A link from node i to node j is denoted as link (i, j). For link (i, j), node i is called the *initial* (or *upstream*) node, and node j is called the *terminal* (or *downstream*) node. We also indicate a link by the sequential numbers from 1 to L allocated to all the links in the set L. If a link has node i as its initial node, we say that the link is incident *from* node i; whereas if a link has node i as its terminal node, we say that the link is incident *to* node i. The set of links incident from node i is denoted by $O(i)$, and the set of links incident to node i is denoted by $I(i)$.

The structure of a network is represented by a reduced node-link incidence matrix $\mathbf{A}$, which is an (N-1) × L matrix *which is achieved by removing the row corresponding to the unique origin* from a standard incidence matrix whose (n, a) element is 1 if link a is incident from node n, -1 if link a is incident to node n, zero otherwise [The reason why we should use this particular type of *reduced* incidence matrix is discussed in Akamatsu (2000)]. It is convenient to "split" the matrix $\mathbf{A}$ into a pair of matrices, $\mathbf{A}_-$ and $\mathbf{A}_+$, which are defined as follows: $\mathbf{A}_-$ is a matrix obtained by letting all the $+1$ elements of $\mathbf{A}$ be zero (*ie* the (n, a) element is -1 if node n is the downstream node of link a, zero otherwise); similarly, $\mathbf{A}_+$ is a matrix obtained by letting all the -1 elements of $\mathbf{A}$ be zero (*ie* the (n, a) element is $+1$ if node n is the upstream node of link a, zero otherwise); it is obvious that $\mathbf{A} = \mathbf{A}_- + \mathbf{A}_+$ holds. Thus $\mathbf{A}_- = \text{Min}(0, \mathbf{A})$ and $\mathbf{A}_+ = \text{Max}(0, \mathbf{A})$.

1.2. Dynamic User Equilibrium Assignment

In this paper, we deal with the dynamic user equilibrium (DUE) assignment. The DUE is defined as the state where at each time, no user can reduce his/her travel time by changing his/her route unilaterally (see Smith (1993), Kuwahara and Akamatsu (1993), Heydecker and Addison (1996)). For a link model in our dynamic assignment, we employ a First-in-First-Out (FIFO) principle and the deterministic point (vertical) queue concept in which a vehicle has no physical length: we assume that the arrival flow on each link leaves that link after the free-flow travel time if no queue is present on the link, otherwise it leave the link at the capacity flow rate after incurring some delay in the queue. This has the advantage of supporting clear-cut analysis whilst leading to results that are typical of those for all models with well-defined link flow capacities.

Previous studies on DUE assignment have shown that assignment with any plausible link model can be decomposed with respect to the departure time from a single origin provided that the OD pattern is one-to-many: we can consider the assignments sequentially in the order of departure from the single origin. The reason why the decomposition is valid for the DUE assignment can be seen from the following facts that hold for the DUE state with any link traffic model that respects the FIFO discipline and has satisfactory causality (for details, see Kuwahara and Akamatsu (1993), Akamatsu and Kuwahara (1994), Heydecker and Addison (1996), Heydecker and Addison (1998)): (1) the users who depart their origin at the same time, regardless of their routes, have the same arrival time at any node that they pass through in common on the way to their destinations; (2) the order of departure from the origin must be maintained at any intermediate node; (3) [from the properties (1) and (2)] we can define the unique equilibrium arrival time at each node for each departure time from the origin; (4) [from (1) and (2) together with the FIFO and the causality properties of the link traffic model] the travel time experienced by the vehicle that departs from an origin at time s is independent of the flows of the vehicles that depart from the origin after time s.

2. DETECTION OF CAPACITY PARADOXES

Under the assumptions described in the previous section, we will establish a condition for occurrence of the capacity paradox (*ie* we will present the method for detecting the links whose increase (decrease) in capacity can worsen (respectively, improve) the total travel cost of a network). In the present paper, we confine our analysis to "*saturated networks*" which satisfy the following two conditions: (a) there are inflows on all links of the network, and (b) there are queues on all links of the network.   The first condition (a) is not very restrictive, because we can constitute the networks satisfying this condition after knowing the set of links with strictly positive flows.   Furthermore, if a link carries flow during some but not all of the study period, then we can apply the present analysis separately to intervals during which the set of links that do carry flow is constant. The second condition (b) seemingly restricts direct applications of the theory; nevertheless, we employ this condition here   because the analysis identifies the essential properties of the paradox (and the DUE assignment). Elsewhere, (Akamatsu and Heydecker, 2001), we show how the theory based on the present assumption of "saturated networks" provides a valuable stepping-stone for the analysis of "non-saturated networks" where the condition (b) is relaxed.

We first show the formulation, solutions, and an algorithm for the DUE assignment in section **2.1**; and then derive the necessary and sufficient condition for the capacity paradox to occur in section **2.2**.

2.1. Dynamic Equilibrium Patterns on Saturated Networks
(1) Formulation

The DUE assignment on a network with a one-to-many OD pattern can be decomposed with respect to the origin departure-time as mentioned in Section **1**. In the decomposed formulation with origin departure time s, two kinds of variables, (y_{ij}^s, τ_i^s), play a central role: τ_i^s is the earliest arrival time at node i for a vehicle that departs from origin o at time s; y_{ij}^s is the

link flow rate *with respect to s*, that is,

$$y_{ij}^s \equiv \frac{dF_{ij}(\tau_i^s)}{ds} = \frac{dF_{ij}(\tau_i^s)}{d\tau_i^s}\frac{d\tau_i^s}{ds} = \lambda_{ij}(\tau_i^s) \cdot \frac{d\tau_i^s}{ds},$$

where $F_{ij}(t)$ denotes the cumulative number of vehicles that have entered link (i, j) at time t, and $\lambda_{ij}(t)$ is the flow rate of link (i, j) at time t defined as $dF_{ij}(t)/dt$. In addition, we denote by $Q_{od}(s)$ the number of vehicles with destination d that have departed from origin o at time s: this is the cumulative OD demand by *departure-time*.

In the DUE state, each user chooses the route that has minimum (*ex post*) travel time over the network. In other words, links with positive inflows should be on the minimum cost paths. In our saturated networks, all the links have positive inflows, and therefore the minimum cost path condition for users with origin departure-time s is written as

$$\tau_j^{s*} - \tau_i^{s*} = c_{ij}^{s*} \qquad \qquad \forall(i, j) \in L \qquad (1)$$

where c_{ij}^{s*} denotes the travel time in equilibrium for users who depart the origin at time s (and arrive at the entrance of the link at time τ_i^{s*}) to traverse link (i, j); ie $c_{ij}^{s*} \equiv c_{ij}(\tau_i^{s*})$. Because equation (1) holds for any departure time s, taking the derivative with respect to s, we have

$$\dot{\mathbf{c}}^*(s) + \mathbf{A}^T \dot{\boldsymbol{\tau}}^*(s) = 0 \qquad \qquad \forall s \qquad (2)$$

where $\dot{\mathbf{c}}^*(s)$ is an L dimensional column vector with elements dc_{ij}^{s*}/ds, and $\dot{\boldsymbol{\tau}}^*(s)$ is an N-1 dimensional column vector with elements $d\tau_i^{s*}/ds$.

The deterministic queueing link model that we use in the present analysis has the following characteristics. The free-flow travel time for each link (i, j) is m_{ij}, and the capacity (maximum outflow rate) is μ_{ij}. In this case, the cost of travel on the link is given by

$$c_{ij}(t) = m_{ij} + \frac{X_{ij}(t + m_{ij})}{\mu_{ij}},$$

where $X_{ij}(t)$ is the amount of traffic in the queue at time t and satisfies

$$\frac{dX_{ij}(t + m_{ij})}{dt} = \begin{cases} \lambda_{ij}(t) - \mu_{ij} & \text{if } X_{ij}(t + m_{ij}) > 0 \text{ or } \lambda_{ij}(t) - \mu_{ij} > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Thus, the rate of change in travel time of link (i, j) at time t, when there is a queue, is given by

$$\frac{dc_{ij}(t)}{dt} = \frac{\lambda_{ij}(t)}{\mu_{ij}} - 1,$$

and therefore, the rate of change in the travel time *for users with origin departure time s*, dc_{ij}^s / ds, can be represented as a function of y_{ij}^s and τ_i^s:

$$\frac{dc_{ij}^{s*}}{ds} = \frac{dc_{ij}(\tau_i^{s*})}{d\tau_i^{s*}} \frac{d\tau_i^{s*}}{ds} = \frac{y_{ij}^{s*}}{\mu_{ij}} - \frac{d\tau_i^{s*}}{ds} \qquad \forall (i, j) \qquad (3a)$$

or equivalently

$$\dot{\mathbf{c}}^*(s) = \mathbf{M}^{-1} \mathbf{y}^*(s) - \mathbf{A}_+^T \dot{\boldsymbol{\tau}}^*(s) \qquad \forall s \qquad (3b)$$

where $\mathbf{M}$ is an L by L diagonal matrix whose a th diagonal element represents the capacity of link a, $\mathbf{y}(s)$ is an L dimensional column vector with elements y_{ij}^s.

Substituting (3) into (2), and rearranging yields the DUE condition for this travel-time model

$$\mathbf{y}^*(s) = -\mathbf{M}\mathbf{A}_-^T \dot{\boldsymbol{\tau}}^*(s) \qquad \forall s \qquad (4)$$

where $\mathbf{y}^*$ is a DUE assignment. Thus we can characterise a dynamic equilibrium assignment equivalently by specifying either flows $\mathbf{y}^*(s)$ or rate of change of travel time $\dot{\boldsymbol{\tau}}^*(s)$.

In addition to this condition above, we have the flow constraints that consist of the FIFO condition for each link and the flow conservation at each node over a network, which reduce to the following equations (for the derivation, see Kuwahara and Akamatsu (1993), Akamatsu and Kuwahara (1994)):

$$-\mathbf{A}\,\mathbf{y}(s) = \dot{\mathbf{Q}}(s) \qquad \forall s, \qquad (5)$$

where $\dot{\mathbf{Q}}(s)$ is defined as an (N-1) dimensional vector with elements $dQ_{od}(s)/ds$ (given).

Combining (5) with (4), we see that the DUE solution (ie $\dot{\boldsymbol{\tau}}^*(s)$) is governed by

$$\left(\mathbf{A}\mathbf{M}\mathbf{A}_-^T\right) \dot{\boldsymbol{\tau}}^*(s) = \dot{\mathbf{Q}}(s) \qquad \forall s. \qquad (6)$$

It is worth mentioning that the equation (6) expressing the DUE condition in a saturated

network is almost the same as the fundamental equation for an electrical circuit with a certain kind of devices called "unistar" (see, for example, Dodd(1967)). In light of the fact that a large number of studies have been made on the electrical circuit theory over a century, it may be useful to investigate the correspondence between the circuit theory and the dynamic traffic assignment. Some exploration is made in this direction in Section **3**.

(2) Solutions

Equation (6) shows that the DUE assignment has a unique solution if the rank of the matrix $\mathbf{V} \equiv \mathbf{AMA}_{-}^{T}$ is N-1. As shown in Akamatsu (2000), the rank is always N-1 in our reduced incidence matrix in which an origin is employed as a reference node and the corresponding row is eliminated from a standard incidence matrix. [Note that the value of $d\tau_{o}(s)/ds = 1$ for an origin node o from the definition of $\tau_{i}(s)$, although the equation (6) does not include the corresponding variable.] Thus, the equilibrium solution $\dot{\boldsymbol{\tau}}^{*}(s)$ is given by the following formula:

$$\dot{\boldsymbol{\tau}}^{*}(s) = \mathbf{V}^{-1}\dot{\mathbf{Q}}(s). \tag{7a}$$

Finally, the equilibrium link flows $\mathbf{y}^{*}(s)$ can be obtained by substituting (7a) into (4): thus

$$\mathbf{y}^{*}(s) = -\mathbf{MA}_{-}^{T}\left(\mathbf{AMA}_{-}^{T}\right)^{-1}\dot{\mathbf{Q}}(s) \tag{7b}$$

(3) An Efficient Algorithm

Equation (6) or (7) can be solved numerically by an appropriate algorithm for solving a system of linear equations even in large-scale networks. However, it is worth mentioning that the equilibrium solution can also be obtained by simple *manual calculations*, which give us insights into the solution. The method is also far more efficient than applying standard algorithms for solving linear equations directly. In what follows, we first explain the basic idea, and then show the formalised procedure. [For notational brevity, we suppress dependence on departure time *s*].

Suppose we know the value of $\dot{\tau}_{j}^{*}$. Then we can obtain the equilibrium flow y_{ij}^{*} on link (i, j) from (4). Summing (4) with respect to the links incident to node *j*, we have that f_{j}^{n}, the sum of in-flows into node *j*, is given by

$$f_j^n = \sum_{(i,j)\in I(j)} \mu_{ij}\, \dot{\tau}_j^*, \tag{8}$$

On the other hand, equation (5) shows that the sum of in-flows f_j^n equals to sum of out-flows f_j^x from node j. Together with (8), this yields

$$f_j^x = \sum_{(i,j)\in I(j)} \mu_{ij}\, \dot{\tau}_j^* . \tag{9}$$

This implies that conversely we can get the value of $\dot{\tau}_j^*$ by knowing the sum of out-flows from node j :

$$\dot{\tau}_j^* = f_j^x \Big/ \sum_{(i,j)\in I(j)} \mu_{ij} . \tag{10}$$

Note here that sum of out-flows from a pure destination d, f_d^x, is known in advance as the OD flow rate $q_{od}(s)$. Hence, we first get the value of $\dot{\tau}_d^*$ at the pure destination:

$$\dot{\tau}_d^* = q_{od} \Big/ \sum_{(i,d)\in I(d)} \mu_{id}$$

Substituting this into (4) then gives flow on each link (i, d) incident to the pure destination:

$$y_{id}^* = \mu_{id}\tau_d^* \qquad \text{for all } (i, d) \in I(d)$$

We repeat the same procedure if there are several pure destinations. These, in turn, give sum of out-flows f_j^x from an appropriate node j (where all the emanating links are incident to pure destinations), and substituting it into (10) yields the value of $\dot{\tau}_j^*$ at the node. Thus we repeat the procedure tracking from an arbitrary pure destination node in a "backward" manner until an origin node have been reached. This backward tracking procedure works through the nodes whose out-flows have been computed, and yields the equilibrium solution $(\dot{\tau}_i^*, y_{ij}^*)$ for all nodes and links in a network.

Example: Consider the network shown in Figure 1, where node 1 is the unique origin, nodes 2 and 3 are destinations, and the maximum outflow rate (capacity) of link a ($a = 1, 2, 3$) is given by μ_a. First we consider conservation of flow at a pure destination, node 3. The outflow from node 3 is given by

$$f_3^x = \dot{Q}_{13}(s) .$$

On the other hand, the sum of inflows into node 3 is

$$f_3^n = y_2 + y_3 = (\mu_2 + \mu_3)\,\dot\tau_3.$$

By conservation of traffic, the inflow f_3^n is equal to the outflow f_3^x, so we have

$$\dot\tau_3^* = f_3^n /(\mu_2 + \mu_3) = f_3^x /(\mu_2 + \mu_3) = \dot Q_{13}(s)/(\mu_2 + \mu_3).$$

Substituting this into (4), we can determine the flows on each of the links incident to node 3:

$$y_2^* = \mu_2\,\dot\tau_3^* = \frac{\mu_2}{\mu_2 + \mu_3}\dot Q_{13}(s), \quad y_3^* = \mu_3\,\dot\tau_3^* = \frac{\mu_3}{\mu_2 + \mu_3}\dot Q_{13}(s)$$

Having obtained each outflow from node 2 (*ie* the flow on each link exiting node 2), we then proceed backward by applying the corresponding argument to node 2. The sum of outflows from node 2 is

$$f_2^x = y_3^* + \dot Q_{12}(s) = \frac{\mu_3}{\mu_2 + \mu_3}\dot Q_{13}(s) + \dot Q_{12}(s),$$

which is equal to the sum of inflows:

$$f_2^n = y_1^* = \mu_1\,\dot\tau_2^*.$$

Hence,

$$\dot\tau_2^* = f_2^n / \mu_1 = f_2^x / \mu_1 = \frac{1}{\mu_1}\left(\frac{\mu_3}{\mu_2 + \mu_3}\dot Q_{13}(s) + \dot Q_{12}(s)\right),$$

and $$y_1 = \mu_1\,\dot\tau_2^* = \frac{\mu_3}{\mu_2 + \mu_3}\dot Q_{13}(s) + \dot Q_{12}(s).$$

Tracking further backward to node 1, we can examine that the link flows y_1 and y_2 obtained above are indeed consistent with the flow conservation at node 1:

$$f_1^x = y_1^* + y_2^* = \left(\frac{\mu_3}{\mu_2 + \mu_3}\dot Q_{13}(s) + \dot Q_{12}(s)\right) + \left(\frac{\mu_2}{\mu_2 + \mu_3}\dot Q_{13}(s)\right)$$

$$= \dot Q_{13}(s) + \dot Q_{12}(s) = f_1^n.$$

In order for the procedure above to give a consistent solution, it is required that each node is treated only when all those downstream have been, which can be achieved only if the network contains no loops. Fortunately, the condition is guaranteed for the DUE flow pattern because we consider only links that have non-zero flows. Thus, the formalised procedure for solving the DUE assignment under a saturated network can be summarised as follows:

Procedure DUE_SN($\mathbf{A}, \mu, \dot{\mathbf{Q}}, \dot{\tau}, \mathbf{y}$)

Initialization:

 LIST := a list where all pure destination nodes are placed in an arbitrary order;

 for each node i in N *do begin*

  Lout $(i) := |O(i)|$;

  If node $i \in S$ then $z_i := \dot{Q}_{oi}$ else $z_i := 0$

 end

Calculation of $\dot{\tau}^*$ and $\mathbf{y}^*$:

 while LIST is not null *do begin*

  j := the first element of LIST;

  $\dot{\tau}^*_j := z_j / \sum_{(i,j) \in I(j)} \mu_{ij}$;

  Remove node j from LIST;

  for each link (i, j) in $I(j)$ *do begin*

   $y^*_{ij} := \mu_{ij}\, \dot{\tau}^*_j$;

   $z_i := z_i + y^*_{ij}$;

   Lout$(i) :=$ Lout$(i) - 1$;

   If Lout$(i) = 0$ then place node i on the bottom of LIST

  end

 end.

2.2. Paradox Occurrence Conditions

To consider the "capacity paradox", we employ the total travel time C for the users departing from an origin from time 0 to T as an index to measure the efficiency of the network flow pattern:

$$C \equiv \int_0^T \mathbf{y}(s)^T \mathbf{c}(s)\, ds = \int_0^T \dot{\mathbf{Q}}(s)^T (\tau(s) - s\mathbf{1})\, ds \tag{11}$$

We then refer to the situation as a "paradox" if increasing (decreasing) the capacity of a certain link causes the increase (decrease) of C (ie the paradox occurs if and only if $\partial C / \partial \mu_a \geq 0$).

From the definition of C in (11), $\partial C / \partial \mu_a$ can be represented as

$$\frac{\partial C}{\partial \mu_a} = \int_0^T \dot{\mathbf{Q}}(s)^T \left(\int_0^s \frac{\partial \dot{\boldsymbol{\tau}}^*(t)}{\partial \mu_a} dt \right) ds. \tag{12}$$

That is, deriving the sensitivity of the DUE solution $\dot{\boldsymbol{\tau}}^*(s)$ with respect to the change in capacity of link a leads to an explicit formula for $\partial C / \partial \mu_a$. In the derivation of the sensitivity formula below, we denote by $\mathbf{M}(\boldsymbol{\mu})$ a diagonal matrix whose diagonal elements are link capacity vector $\boldsymbol{\mu} = [\cdots, \mu_a, \cdots]$, and by $\mathbf{V}(\boldsymbol{\mu})$ a matrix $\mathbf{A}\mathbf{M}(\boldsymbol{\mu})\mathbf{A}_-^T$. For notational brevity, we suppress dependence on departure time s because all equations used here take the same form regardless of departure times.

Let us first consider the two equilibrium solutions, $\dot{\boldsymbol{\tau}}^*(\boldsymbol{\mu})$ and $\dot{\boldsymbol{\tau}}^*(\boldsymbol{\mu} + \Delta\boldsymbol{\mu})$, where here the dependence of $\dot{\boldsymbol{\tau}}^*$ on the capacity patterns $\boldsymbol{\mu}$ and $\boldsymbol{\mu} + \Delta\boldsymbol{\mu}$ is shown explicitly. From (6), the two solutions are governed by the following equations:

$$\mathbf{V}(\boldsymbol{\mu}) \, \dot{\boldsymbol{\tau}}^*(\boldsymbol{\mu}) = \dot{\mathbf{Q}}(s), \tag{13a}$$

$$[\mathbf{V}(\boldsymbol{\mu}) + \mathbf{V}(\Delta\boldsymbol{\mu})] \, \dot{\boldsymbol{\tau}}^*(\boldsymbol{\mu} + \Delta\boldsymbol{\mu}) = \dot{\mathbf{Q}}(s), \tag{13b}$$

where we used the fact that $\mathbf{V}(\boldsymbol{\mu} + \Delta\boldsymbol{\mu}) = \mathbf{V}(\boldsymbol{\mu}) + \mathbf{V}(\Delta\boldsymbol{\mu})$ holds for any capacity increase patterns $\boldsymbol{\mu} + \Delta\boldsymbol{\mu}$ because $\mathbf{V}(\boldsymbol{\mu}) = \mathbf{A}\mathbf{M}(\boldsymbol{\mu})\mathbf{A}_-^T$ is linear in $\boldsymbol{\mu}$. We then compare the solutions: subtracting (13a) from (13b), we have

$$\begin{aligned} \dot{\boldsymbol{\tau}}^*(\boldsymbol{\mu} + \Delta\boldsymbol{\mu}) - \dot{\boldsymbol{\tau}}^*(\boldsymbol{\mu}) &= -\mathbf{V}^{-1}(\boldsymbol{\mu}) \, \mathbf{V}(\Delta\boldsymbol{\mu}) \, \dot{\boldsymbol{\tau}}^*(\boldsymbol{\mu} + \Delta\boldsymbol{\mu}) \\ &= -\mathbf{V}^{-1}(\boldsymbol{\mu}) \, \mathbf{V}(\Delta\boldsymbol{\mu}) \, [\mathbf{V}(\boldsymbol{\mu}) + \mathbf{V}(\Delta\boldsymbol{\mu})]^{-1} \, \dot{\mathbf{Q}}(s). \end{aligned} \tag{14}$$

Consider the case where $\Delta\boldsymbol{\mu} = [0,...,0,\Delta\mu_a,0...,0]$. Dividing both sides of equation (14) by $\Delta\mu_a$, using the identity $\mathbf{V}(\Delta\boldsymbol{\mu})/\Delta\mu_a = \mathbf{A}\,\mathbf{I}_a\mathbf{A}_-^T$, and taking the limit as $\Delta\mu_a \to 0$, we obtain

$$\frac{\partial \dot{\boldsymbol{\tau}}^*(\boldsymbol{\mu})}{\partial \mu_a} = -\mathbf{V}^{-1}(\boldsymbol{\mu}) \, \mathbf{A}\,\mathbf{I}_a\mathbf{A}_-^T \, \mathbf{V}^{-1}(\boldsymbol{\mu}) \, \dot{\mathbf{Q}}(s), \tag{15a}$$

where $\partial \dot{\boldsymbol{\tau}}^{*}(\boldsymbol{\mu})/\partial \mu_{a}$ is an N-1 dimensional column vector with elements $\partial \dot{\tau}_{i}^{*}(\boldsymbol{\mu})/\partial \mu_{a}$, and $\mathbf{I}_{a}$ is an L by L matrix whose a th diagonal element is 1 and all other elements are 0.

We can obtain a simple expression for the typical element of (15a), by noticing that $\mathbf{A}\,\mathbf{I}_{a}\mathbf{A}_{-}^{T}$ has at most two non-zero elements: denote the upstream node and the downstream node of link a by i_{a} and j_{a}, respectively, then the element in row i_{a} and column j_{a} is -1, and the element in row j_{a} and column i_{a} is 1. This means that element k in (15a) is given by

$$\frac{\partial \dot{\tau}_{k}^{*}(s)}{\partial \mu_{ij}} = -(v_{kj}^{-1} - v_{ki}^{-1})\,\dot{\tau}_{j}^{*}(s)\,, \tag{15b}$$

where v_{ij}^{-1} is (i, j) element of $\mathbf{V}^{-1}$. We note that although (15a) might appear to be complicated, the sensitivity of $\dot{\boldsymbol{\tau}}^{*}(s)$ can be calculated easily by an appropriate procedure exploiting DUE_SN in section **2.1**.

We are now in a position to present the necessary and sufficient condition for the paradox to occur: substituting the sensitivity formula (15) into (12), we have the following proposition.

Proposition 1. *The capacity paradox in a saturated network occurs if and only if*

$$\frac{\partial C}{\partial \mu_{a}} = -\int_{0}^{T} \dot{\mathbf{Q}}(s)^{T}\,\mathbf{V}^{-1}\mathbf{A}\mathbf{I}_{a}\mathbf{A}_{-}^{T}\mathbf{V}^{-1}\{\mathbf{Q}(s) - \mathbf{Q}(0)\}\,ds \geq 0\,, \tag{16a}$$

or equivalently

$$\frac{\partial C}{\partial \mu_{ij}} = -\int_{0}^{T} \sum_{k} \dot{Q}_{ok}(s)\,(v_{kj}^{-1} - v_{ki}^{-1})\,(\tau_{j}^{*}(s) - \tau_{j}^{*}(0))\,ds \geq 0. \tag{16b}$$

Example: Consider the network shown in Figure 1, where node 1 is the unique origin; nodes 2 and 3 are destinations; the maximum departure rate (capacity) of link a ($a = 1, 2, 3$) is given by μ_{a}.

Fig. 1. An Example Network with a Single Origin and Two Destinations

The reduced incidence matrix $\mathbf{A}$ and the matrix $\mathbf{A}_{-}$ for this network are given by

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}, \quad \mathbf{A}_- = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}.$$

Hence,

$$\mathbf{V} \equiv \mathbf{A}\mathbf{M}\mathbf{A}_-^T = \begin{bmatrix} \mu_1 & -\mu_3 \\ 0 & \mu_2 + \mu_3 \end{bmatrix}, \quad \mathbf{V}^{-1} = \begin{bmatrix} \dfrac{1}{\mu_1} & \dfrac{\mu_3}{\mu_1(\mu_2 + \mu_3)} \\ 0 & \dfrac{1}{\mu_2 + \mu_3} \end{bmatrix},$$

$$\mathbf{A}\mathbf{I}_1\mathbf{A}_-^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{A}\mathbf{I}_2\mathbf{A}_-^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}\mathbf{I}_3\mathbf{A}_-^T = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

Substituting these into (16) yields

$$\frac{\partial C}{\partial \mu_1} = -\frac{1}{\mu_1^{\,2}} \int_0^T \left(Q_{12}(s) + \frac{\mu_3}{\mu_2 + \mu_3} Q_{13}(s) \right) \dot{Q}_{12}(s)\, ds \qquad (17a)$$

$$\frac{\partial C}{\partial \mu_2} = -\frac{1}{(\mu_2 + \mu_3)^2} \int_0^T \left(\frac{\mu_3}{\mu_1} \dot{Q}_{12}(s) + \dot{Q}_{13}(s) \right) Q_{13}(s)\, ds \qquad (17b)$$

$$\frac{\partial C}{\partial \mu_3} = \frac{1}{(\mu_2 + \mu_3)^2} \int_0^T \left(\dot{Q}_{13}(s) - \frac{\mu_2}{\mu_1} \dot{Q}_{12}(s) \right) Q_{13}(s)\, ds \qquad (17c)$$

From these, we see that the paradox does not occur when we change the capacity of either of links 1 or 2 because both (17a) and (17b) always have negative values; however, it is seen from (17c) that the paradox will certainly occur in respect of capacity variations for link 3 if $\dot{Q}_{13}(s) \geq (\mu_2 / \mu_1)\, \dot{Q}_{12}(s)$ holds for all s.

We can understand this as follows. If the capacity of link 3 is increased, then it will attract from link 2 to link 3 and hence to link 1 some traffic that is destined for node 3. This reduction in flow on link 2 will reduce the cost of travel to node 3, which in equilibrium is identical on the two routes {2} and {1, 3}. However, the increase in flow on link 1 will increase the cost of travel to node 2. Whether or not these changes in cost of opposite sign result in an increase in total travel cost C for a marginal increase in capacity of link 3 is determined by the sign of $\int_0^T [\dot{Q}_{13}(s) - (\mu_2 / \mu_1)\, \dot{Q}_{12}(s)]\, Q_{13}(s)\, ds$.

This example can be contrasted with Braess' paradoxical network for static equilibrium

assignment. In Braess' network, a marginal increase in capacity of a link causes a marginal increase in the total cost of travel at equilibrium: because there is a single origin-destination pair, the cost of travel is the same for all travellers, so a capacity increase leaves all travellers worse off. However, in the present example, the increase in the total cost of travel arises in cases where an increase in travel cost for one zone outweighs the decrease in cost enjoyed by travellers to another one who benefit from use of the link that is improved.

3. GRAPH THEORETIC INTERPRETATIONS

In the previous section, we established a necessary and sufficient condition for a capacity paradox to occur in a saturated network. This section identifies a connection between the condition and structural properties of networks. For this purpose, we interpret the condition from a graph-theoretic point of view: we first show graph theoretic expressions of the DUE solution in **3.1**, and then derive another expression for the paradox occurrence condition in **3.2**.

3.1 Graph Theoretic Representation of Equilibrium Solutions

Consider a directed graph $\Gamma(N, L)$ where each directed link (i, j) in link set L has link weight b_{ij}. We represent the structure of $\Gamma(N, L)$ by an N by N matrix **B** in the following manner: the entry in the i th row and j th column is $-b_{ij}$ if link (i,j) exists in $\Gamma(N, L)$, the i th diagonal entry (i.e. the entry in row i and column i) is $\sum_{(k,i) \in I(i)} b_{ki}$, and all other elements are zero. Note that this matrix is very similar to the (N-1) by (N-1) matrix $\mathbf{V} \equiv \mathbf{AMA}_{-}^{T}$ in the previous section.

It is known in graph theory that several properties of this particular matrix are related to directed spanning trees in $\Gamma(N, L)$. [A directed spanning tree with a root node i in $\Gamma(N, L)$ is defined as the set of directed links in L such that node i has only links incident from it (i.e. node i has no link incident to it), and every node in N excluding of node i has just one link incident to it.]   In particular, the following lemma is useful for our analyses:

Lemma 3.1. *Suppose that a directed graph* $\Gamma(N, L)$ *has no self-loop link of the form* $a = (i, i)$, *and let* $S(i)$ *be the set of directed spanning trees in* $\Gamma(N, L)$ *with root node i. Then the cofactor* B_{ii} *of* $\mathbf{B}$ *for* $\Gamma(N, L)$ *is represented by*

$$B_{ii} = \sum_{T \in S(i)} \prod_{(p,q) \in T} b_{pq}. \tag{18}$$

Proof : *see Tutte (1948), and Kajitani et al (1982).*

For the convenience of applying this lemma to our present study, we represent the DUE condition (6) in the following form:

$$\mathbf{B}\,\mathbf{x}(s) = \mathbf{0} \tag{19}$$

where $\mathbf{x}(s) \equiv \begin{bmatrix} \dot{\boldsymbol{\tau}}(s) \\ 1 \end{bmatrix}$, $\mathbf{B} \equiv \begin{bmatrix} \mathbf{V} & -\dot{\mathbf{Q}}(s) \\ \boldsymbol{\beta} & \alpha \end{bmatrix}$, $\boldsymbol{\beta}$ is an N-1 dimensional row vector whose j th

element is $\beta_j \equiv -\sum_{i=1}^{N-1} v_{ij}$, and $\alpha \equiv \sum_{k=1}^{N-1} \dot{Q}_{ok}(s)$. The last equation in (19) thus corresponds to the

DUE condition at the origin node o that we deleted because of its redundancy in Section **2**. Note that the graph $\Gamma(N, L)$ corresponding to the matrix $\mathbf{B}$ of this structure is formed by adding "return links" for each destination node to the original network; the return link (d, o) returns directly from destination node d to the origin node o, and has link weight $\dot{Q}_{od}(s)$.

From Cramer's rule, the solution of the linear equation (19) is given by

$$\dot{\tau}_i^*(s) = B_{Ni} / B_{NN} \quad \text{for} \quad i=1,2,...N. \tag{20}$$

Note here that $B_{Ni} = B_{ii}$ $(i=1,2,...N)$ holds for the matrix $\mathbf{B}$ defined in (19), because $\mathbf{B}$ satisfies

$$\sum_{k=1, k \neq i}^{n} b_{kj} = -b_{ij} \quad \text{for} \quad i, j=1,2,...N. \tag{21}$$

From these results and Lemma 3.1, we have the following theorem:

Theorem 3.1. *The DUE solution* $\dot{\tau}_i^*(s)$ *at node i in a saturated network can be expressed in terms of directed spanning trees in* $\Gamma(N,L)$:

$$\dot{\tau}_i^*(s) = B_{ii} / B_{NN} = \frac{\sum\limits_{T \in S(i)} \prod\limits_{(p,q) \in T} \mu_{pq}}{\sum\limits_{T \in S(N)} \prod\limits_{(p,q) \in T} \mu_{pq}} \qquad \text{for} \quad i=1,2,...N. \qquad (22)$$

In order to explore the properties of the DUE solution, it is convenient to represent the solution in terms of directed paths rather than directed spanning trees. Applying appropriate decompositions to the directed spanning trees in (22), the formula can be further transformed into the expression in terms of paths:

Theorem 3.2. *The DUE solution* $\dot{\tau}_i^*(s)$ *at node i in a saturated network can be represented in terms of directed paths in* $P(i,k)$ *for all* $k \in N$:

$$\dot{\tau}_i^*(s) = \alpha_i(s) + \sum_{\substack{k \in N \\ k \neq i}} \alpha_k(s) \sum_{r \in P(i,k)} \prod_{(p,q) \in r} \alpha_{pq} \qquad (23)$$

where $P(i, k)$ is the set of directed paths from node i to node k; $\alpha_k(s)$ and α_{pq} are, respectively, defined as

$$\alpha_k(s) \equiv \dot{Q}_{ok}(s) / \sum_{(\ell,k) \in I(k)} \mu_{\ell k} , \qquad (24a)$$

$$\alpha_{pq} \equiv \mu_{pq} / \sum_{(m,p) \in I(p)} \mu_{mp} . \qquad (24b)$$

Proof*: see **Appendix A**.*

Although the proof of the equation (23) is somewhat lengthy, the equation tell us a simple fact: the DUE solution $\dot{\tau}_i^*(s)$ at node i can be decomposed into two terms, the first term reflecting the effect of flow destined to node i, and the second term consisting of the summation with respect to destinations that are reachable from node i. This leads immediately to the following result:

Corollary. *The DUE solution* $\dot{\tau}_i^*(s)$ *at node i in a saturated network is affected only by flows to destinations that are reachable from node i* (*ie the destinations that are downstream of node i). Thus variations in flows to destinations that are upstream of node i do not affect* $\dot{\tau}_i^*(s)$.

This result is remarkable and might be considered to be counterintuitive from a phenomenological point of view: traffic arriving at a certain destination will have shared capacity of links in the network with other traffic travelling to destinations upstream of it, so that variations in traffic destined to those upstream nodes might be expected to affect the cost of travel. However, this corollary shows that this is not the case and that rather the inter-dependence of costs on demands is strictly the other way round. That this result holds exactly depends on the deterministic queuing model that is used in the present analysis, but similar effects can be expected when other traffic models that respect capacity limitations are adopted.

The nature of this effect can be understood from a mechanistic point of view as follows. In the present analysis we suppose that queues are present on all links of the network. Because of use of the deterministic queueing traffic model, this means that the link outflows are each determined exactly as the corresponding capacity. As a direct consequence of this, variations in demand for travel to the different destinations in the network will result in variations in the composition of the link outflows but not in their total flow rate. Accordingly, variations in flows to destinations upstream of a node i can affect the onwards flows that pass those destinations and hence enter links $(l, i) \in \mathrm{I}(i)$. However, because by supposition there are queues on the links of $\mathrm{I}(i)$, the outflows from these links will be unaffected by marginal variations in the flows upstream. Thus the location at which delays are incurred by traffic that passes through node i can be affected by variations in flows destined to nodes upstream, but the total delay that is incurred will not be.

On the other hand, because the outflow from each link remains equal to its capacity, variations in flows that are destined to nodes downstream of a node i will cause complementary variations in the flows destined to node i itself. Variations in the flow destined to nodes that are downstream of i will then cause variations in the cost of travel to node i itself. This can be seen from a computational point of view in the example application of the algorithm that was presented in Section 2.1 where analysis of the flows and

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travel costs proceeds backwards from the destination towards the origin.

More detailed inspection of each term in (23) reveals several quantitative implications. The first term is proportional to the OD flow to node i, and it vanishes if node i is not a destination node. The second term consists of (a) a summation with respect to destinations, (b) summations with respect to paths from node i to each destination k, and (c) multiplications with respect to links (nodes) in each path from node i to each destination. The summation (a) implies that the second term vanishes if node i is a "pure destination", and that it has larger value as the number (or the OD flows) of destinations reachable from node i increases. Similarly, the summation (b) implies that the second term has larger value as the number of paths from node i to each destination increases. The term (c) is the multiplication of "out/in ratio", α_{pq}, which is the ratio of a link capacity to the (possible) maximum inflow to the link. Hence, the second term grows as the capacity of the links in each path from node i to destinations increases.

3.2 Another Representation of Paradox Occurrence Condition

We have seen in Section **2** that the necessary and sufficient condition for the capacity paradox to occur in a saturated network is given by

$$\frac{\partial C}{\partial \mu_{ij}} = -\int_0^T \sum_k \dot{Q}_{ok}(s) \, (v_{kj}^{-1} - v_{ki}^{-1}) \, (\tau_j^*(s) - \tau_j^*(0)) \, ds \geq 0. \qquad (16b)$$

where v_{ij}^{-1} is (i, j) element of $\mathbf{V}^{-1}$. We note here that v_{ij}^{-1} is equal to $Z_{j(i)} / Z_o$, as implied by equation (A-2) in **Appendix A**. Furthermore, we see from (A-13) that $Z_{i(k)} / Z_o$ is expressed in terms of paths in $P(k, i)$:

$$Z_{i(k)} \Big/ Z_o = \frac{1}{\sum\limits_{(\ell, i) \in I(i)} \mu_{\ell i}} \sum_{r \in P(k,i)} \prod_{(p,q) \in r} \alpha_{pq}$$

$$= \frac{1}{\sum\limits_{(\ell, k) \in I(k)} \mu_{\ell k}} \sum_{r \in P(k,i)} \prod_{(p,q) \in r} \beta_{pq} \qquad (25)$$

where α_{pq} is defined in (24), and β_{pq} is defined as

$$\beta_{pq} \equiv \mu_{pq} / \sum_{(m,q) \in I(q)} \mu_{mq} . \tag{26}$$

Combining these, we have the paradox occurrence condition expressed in terms of directed paths:

Proposition 2. *The capacity paradox occurs in a saturated network if and only if*

$$\frac{\partial C}{\partial \mu_{ij}} = \int_0^T (\dot{\lambda}_i(s) - \dot{\lambda}_j(s))(\tau_j^*(s) - \tau_j^*(0)) \, ds \geq 0 \tag{27}$$

where

$$\dot{\lambda}_i(s) \equiv \sum_{k \in Ni} \alpha_k(s) \sum_{r \in P(k,i)} \prod_{(p,q) \in r} \beta_{pq} \tag{28}$$

$$\tau_i^*(s) - \tau_i^*(0) = \sum_{k \in N} [\int_0^s \alpha_k(t) dt] \sum_{r \in P(i,k)} \prod_{(p,q) \in r} \alpha_{pq} \tag{29}$$

Example: Consider the saturated network shown in Figure 2 (a). Node o is the origin, and nodes b and d are destinations with OD flow rates for the origin departure-time s, $\dot{Q}_{ob}(s)$ and $\dot{Q}_{od}(s)$, respectively. For the expositional brevity, we assume that $Q_{ob}(0) = Q_{od}(0) = 0$. Each link (i,j) in the network has capacity μ_{ij}. The graph $\Gamma(N,L)$ for this network is depicted in Figure 2 (b).

We first calculate the DUE solution by the procedure based on Theorem 3.1, and then show that the procedure can be simplified by using Theorem 3.2.

Fig. 2. An Example Network and the Corresponding Graph Γ

Figure 3 shows all the directed spanning trees in $S(o)$. Hence, the denominator of (22) for this network is

$$\sum_{T \in S(o)} \prod_{(p,q) \in T} \mu_{pq} = \mu_1 \mu_6 (\mu_4 + \mu_5 + \mu_7)(\mu_2 + \mu_3) . \tag{30}$$

Fig. 3. Directed Spanning Trees in $S(o)$

Let us first calculate $\dot{\tau}_d^*$. All the directed spanning trees in $S(d)$ are enumerated in Figure 4, and the numerator of (22) for node d is given by

$$\sum_{T \in S(d)} \prod_{(p,q) \in T} \mu_{pq} = \dot{Q}_{od}\, \mu_1 \mu_6 (\mu_2 + \mu_3). \tag{31}$$

Fig. 4. Directed Spanning Trees in $S(d)$

Substituting (30) and (31) into (22) yields

$$\dot{\tau}_d^*(s) = \frac{\dot{Q}_{od}(s)}{\mu_4 + \mu_5 + \mu_7}. \tag{32}$$

Expressions for $\dot{\tau}_c^*$ and $\dot{\tau}_b^*$ are calculated in a similar manner: all the directed spanning trees in $S(c)$ and $S(b)$ are listed in Figure 5 and Figure 6, respectively; hence, the numerators of (22) for node c and b are given by

$$\sum_{T \in S(c)} \prod_{(p,q) \in T} \mu_{pq} = \dot{Q}_{od}\, \mu_1 \mu_7 (\mu_2 + \mu_3), \tag{33}$$

$$\sum_{T \in S(b)} \prod_{(p,q) \in T} \mu_{pq} = \dot{Q}_{ob}\, \mu_1 \mu_6 (\mu_4 + \mu_5 + \mu_7) + \dot{Q}_{od}\, \mu_1 \mu_6 (\mu_5 + \mu_7), \tag{34}$$

respectively. Substituting these and (30) into (22), we have

$$\dot{\tau}_c^*(s) = \frac{\dot{Q}_{od}(s)}{(\mu_4 + \mu_5 + \mu_7)} \frac{\mu_7}{\mu_6}, \tag{35}$$

$$\dot{\tau}_b^*(s) = \frac{\dot{Q}_{ob}(s)}{\mu_2 + \mu_3} + \frac{\dot{Q}_{od}(s)}{(\mu_4 + \mu_5 + \mu_7)} \left(\frac{\mu_5 + \mu_7}{\mu_2 + \mu_3} \right). \tag{36}$$

Fig. 5. Directed Spanning Trees in $S(c)$

Fig. 6. Directed Spanning Trees in $S(b)$

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It is also possible to obtain the same solution by the method based on Theorem 3.2. We immediately see from (23) that $\dot{\tau}_d^*(s)$ is given as in (32) because node d has no path *to* any other destinations (*ie* node d is a "pure destination"), and hence the second term of (23) has no effect to the value of $\dot{\tau}_d^*(s)$. Node c has only a single path to destination d, {7}, and no path to destination b. The sets of links in $I(c)$ and $I(d)$ are {6} and {4, 5, 7}, respectively. Hence, applying (23) yields (35). Node b has two paths to destination d, {5} and {6,7}. The set of links in $I(b)$ is {2,3}. Applying (23), we get

$$
\begin{aligned}
\dot{\tau}_b^*(s) &= \frac{\dot{Q}_{ob}(s)}{\mu_2 + \mu_3} + \frac{\dot{Q}_{od}(s)}{\left(\mu_4 + \mu_5 + \mu_7\right)} \left\{ \frac{\mu_5}{\mu_2 + \mu_3} + \frac{\mu_6}{\left(\mu_2 + \mu_3\right)} \frac{\mu_7}{\mu_6} \right\} \\
&= \frac{\dot{Q}_{ob}(s)}{\mu_2 + \mu_3} + \frac{\dot{Q}_{od}(s)}{\left(\mu_4 + \mu_5 + \mu_7\right)} \left(\frac{\mu_5 + \mu_7}{\mu_2 + \mu_3} \right).
\end{aligned}
\tag{37}
$$

By using Proposition 2, we can easily examine whether or not the paradox occurs in this network. Let us first calculate $\dot{\lambda}_a(s), \dot{\lambda}_b(s), \dot{\lambda}_c(s)$ and $\dot{\lambda}_d(s)$. Because neither node a nor node b has any path *from* destinations, and node b itself is a destination, it immediately follows from (28) that

$$
\dot{\lambda}_a(s) = 0, \quad \dot{\lambda}_b(s) = \frac{\dot{Q}_{ob}(s)}{\mu_2 + \mu_3}.
\tag{38}
$$

Node c has a single path from destination b, {6}; node d is itself a destination, and has two paths from destination b, {5} and {6,7}, and a single path from destination c, {7}; applying (28), we get:

$$
\dot{\lambda}_c(s) = \frac{\dot{Q}_{ob}(s)}{\left(\mu_2 + \mu_3\right)} \frac{\mu_6}{\mu_6} = \frac{\dot{Q}_{ob}(s)}{\mu_2 + \mu_3},
\tag{39}
$$

$$
\begin{aligned}
\dot{\lambda}_d(s) &= \frac{\dot{Q}_{od}(s)}{\mu_4 + \mu_5 + \mu_7} + \frac{\dot{Q}_{ob}(s)}{\left(\mu_2 + \mu_3\right)} \left\{ \frac{\mu_5}{\mu_4 + \mu_5 + \mu_7} + \frac{\mu_6}{\mu_6} \frac{\mu_7}{\left(\mu_4 + \mu_5 + \mu_7\right)} \right\} \\
&= \frac{1}{\mu_4 + \mu_5 + \mu_7} \left[\dot{Q}_{od}(s) + \dot{Q}_{ob}(s) \left(\frac{\mu_5 + \mu_7}{\mu_2 + \mu_3} \right) \right].
\end{aligned}
\tag{40}
$$

In investigating whether or not the paradox occurs in link (i, j), we should examine the sign of $\dot{\lambda}_i(s) - \dot{\lambda}_j(s)$ because $\tau_j^*(s) - \tau_j^*(0)$ in (29) is always positive. From (38), (39) and (40), we find that $\dot{\lambda}_a(s) - \dot{\lambda}_b(s)$ and $\dot{\lambda}_a(s) - \dot{\lambda}_d(s)$ always have negative value so that the paradox cannot occur in either of links (a, b) and (a, d). Furthermore, $\dot{\lambda}_b(s) - \dot{\lambda}_c(s)$ is always zero so that total travel time is independent of change in capacity of link (b, c). However, $\dot{\lambda}_b(s) - \dot{\lambda}_d(s)$ and $\dot{\lambda}_c(s) - \dot{\lambda}_d(s)$ can be positive if $\dot{Q}_{ob}(s)$ is relatively large in comparison with $\dot{Q}_{od}(s)$:

$$\dot{\lambda}_b(s) - \dot{\lambda}_d(s) = \dot{\lambda}_c(s) - \dot{\lambda}_d(s) = \frac{1}{\mu_4 + \mu_5 + \mu_7}\left[\dot{Q}_{ob}(s)\frac{\mu_4}{\mu_2 + \mu_3} - \dot{Q}_{od}(s)\right] \qquad (41)$$

Therefore, we see that the capacity paradox can occur in links (b, d) and (c, d) if the following condition holds:

$$\int_0^T \left[\frac{\dot{Q}_{ob}(s)}{\mu_2 + \mu_3} - \frac{\dot{Q}_{od}(s)}{\mu_4}\right] Q_{od}(s)\, ds \geq 0. \qquad (42)$$

This condition is identical for both of these links, so if the capacity paradox occurs in either one it will occur in both of them.

3.3. Networks where Paradox Occurrence is Inevitable

Given a capacity pattern $\mathbf{M}$, OD flow patterns $\mathbf{Q}(s)$ for all s in $[0,T]$, and a network structure $\mathbf{A}$, we can detect the paradox occurrence in the network by applying the theory developed so far. However, Proposition 2 in section **3.2**, tells us that knowledge of the network structure $\mathbf{A}$ is sufficient to detect the paradox occurrence in some cases (*i*e neither $\mathbf{M}$ nor $\mathbf{Q}(s)$ is needed): the sign of $\dot{\lambda}_i(s) - \dot{\lambda}_j(s)$ in Proposition 2, which determines the sign of $\partial C / \partial \mu_{ij}$, depends only on path patterns of the network regardless of $\mathbf{M}$ and $\mathbf{Q}(s)$. The following proposition identifies some networks in which we can guarantee no occurrence of the paradox without the information on $\mathbf{M}$ and $\mathbf{Q}(s)$:

Proposition 3.1. *Suppose that we change the capacity of link* (i, j) *on a saturated network with a one-to-many OD demand pattern, and let S be the set of destination nodes in this network. Then the capacity paradox <u>never occurs</u> in respect of this link if the network structure satisfies the following conditions*:

(*a*) *neither of nodes i and j is a destination,*

(*b*) *for each destination node d* ∈ *S, there is no path from node d to node i ,*

(*c*) *for at least one destination node d* ∈ *S, there exists a path from node d to node j .*

Proof: see ***Appendix B***.

Figure 7 shows a simple example of a network that satisfies the conditions (*a*), (*b*) and (*c*) of Proposition 3.1 in respect of link (i, j): neither destination 1 nor 2 has a path to node *i*, and destination 1 has a path to node *j*. From the theorem we know that changing the capacity of link (i, j) *never* causes the paradox regardless of the OD demand pattern and the capacity pattern in the network.

Figure 7. An Example Network for Proposition 3.1 (Capacity Paradox Never Occurs)

Similarly, the following proposition identifies some networks where the paradox occurrence is inevitable for all patterns of **M** and **Q**(*s*):

Proposition 3.2. *Suppose that we change the capacity of link* (i, j) *in a saturated network with a one-to-many OD demand pattern, and let S be the set of destination nodes in this network. Then the capacity paradox <u>always occurs</u> in respect of this link if the network structure satisfies the following conditions*:

(*a*) *neither of nodes i nor j is a destination ,*

(*b*) *for at least one of the* destination *nodes d* ∈ *S, there exists path(s) from node d to node i ,*

(*c*) *for each destination node d* ∈ *S, there is no path from node d to node j other than those that traverses node i.*

Proof: see ***Appendix C.***

Figure 8 shows a simple example of a network that satisfies conditions (*a*), (*b*) and (*c*) of Proposition 3.2: destination 1 has a path to node i, and neither of destinations 1 and 2 has a path that reaches node j without traversing node i. The theorem says that change in the capacity of link (i, j) *necessarily* causes the paradox regardless of the OD demand pattern and the capacity pattern in the network.

Figure 8. An Example Network for Proposition 3.2 (Capacity Paradox Always Occurs)

4. CONCLUDING REMARKS

This paper presents a theory to detect the occurrence of a "capacity paradox" under dynamic user equilibrium (DUE) assignment in a general network with a one-to-many OD demand pattern. Defining the paradox as the situation that increase (decrease) in capacity of a link leads to an increase (decrease) in total travel time over a network, we first derive a necessary and sufficient condition for the paradox to occur in "saturated networks" in which there is a queue on each link. We then give a graph theoretic interpretation of the condition, which enables us to identify network structures in which the paradox always occurs regardless of capacity and demand patterns.

The models that have been used in developing this theory are specific cases of those used in dynamic traffic assignment, so the results will not necessarily hold exactly for other combinations (for example, stochastic assignment principles or other traffic models). However, similar effects will no doubt arise so that in cases where the results presented here indicate a strong occurrence of the cost increasing paradox, one could reasonably expect it to occur to some extent whatever combination of assignment and traffic models is adopted.

The theory presented in this paper is based upon the strong assumption of saturated networks in which congestion is present on every link. We would emphasise, however, that this theory has wider implications and forms the basis of analyses that are applicable to more

general networks in which some but not all links are saturated. In analysing dynamic traffic assignments in non-saturated networks, we can develop ways of reducing them to related saturated ones from which relevant deductions can be made. By this means, we can analyse the properties of non-saturated networks and identify circumstances in which capacity paradoxes will arise. Thus we can establish the more general relevance of the present results. Detailed development and discussion of techniques for this and example applications of the results are the subject of further research that will be reported in due course.

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APPENDIX A: Derivation of Theorem 3.2

In the followings, we suppress dependence on the departure time s, and denote the N th node by subscript o to indicate explicitly that it is the origin node in the original network. Let us begin with a simple observation that each tree in the numerator of (22) includes exactly one return link because only a single link should be incident to the origin node in the tree. From this fact, the numerator of (22) can be decomposed with respect to OD pair:

$$\sum_{T\in S(i)} \prod_{(p,q)\in T} \mu_{pq} = \sum_{k\in N}\dot{Q}_{ok} \sum_{T\in S(i;k,o)} \prod_{\substack{(p,q)\in T \\ (p.q)\neq(k,o)}} \mu_{pq}$$

$$= \dot{Q}_{oi} \sum_{T\in S(i;i,o)} \prod_{\substack{(p,q)\in T \\ (p.q)\neq(i,o)}} \mu_{pq} + \sum_{\substack{k\in N \\ k\neq i}}\dot{Q}_{ok} \sum_{T\in S(i;k,o)} \prod_{\substack{(p,q)\in T \\ (p.q)\neq(k,o)}} \mu_{pq}, \qquad \text{(A-1a)}$$

where $S(i; k, o)$ is the subset of $S(i)$ whose element (each tree) includes link (k, o). Introducing the set of "pseudo-trees", $S(i; (k, o))$, whose element (pseudo-tree) is formed by deleting link (k, o) from the tree in $S(i; k, o)$, we rewrite (A-1a) as the following expression:

$$\sum_{T\in S(i)} \prod_{(p,q)\in T} \mu_{pq} = \dot{Q}_{oi} \sum_{T\in S(i;(i,o))} \prod_{(p,q)\in T} \mu_{pq} + \sum_{\substack{k\in N \\ k\neq i}}\dot{Q}_{ok} \sum_{T\in S(i;(k,o))} \prod_{(p,q)\in T} \mu_{pq}. \qquad \text{(A-1b)}$$

Substituting this into (22), we have

$$\dot{\tau}_i^* = \dot{Q}_{oi} \frac{Z_{i(i)}}{Z_o} + \sum_{\substack{k\in N \\ k\neq i}}\dot{Q}_{ok} \frac{Z_{k(i)}}{Z_o}, \qquad \text{(A-2)}$$

where $Z_{k(i)}$ and Z_o are defined as

$$Z_{k(i)} \equiv \sum_{T\in S(i;(k,o))} \prod_{(p,q)\in T} \mu_{pq}, \qquad \text{(A-3)}$$

$$Z_o \equiv \sum_{T\in S(o)} \prod_{(p,q)\in T} \mu_{pq}. \qquad \text{(A-4)}$$

Next, we transform $Z_{i(i)}$, $Z_{k(i)}$ and Z_o into more convenient representation. We first take notice of the fact that the summation over elements of $S(o)$ in (A-4) can be transformed to a summation with respect to nodes in N. To see this, suppose there are several links (ℓ, i) incident to node i, then $S(o)$ can be decomposed into corresponding subsets, $S(o; \ell, i)$ for each $(\ell, i) \in I(i)$. This decomposition means that (A-4) is represented by

$$Z_o = \sum_{(\ell,i)\in I(i)} \mu_{\ell i} \sum_{T\in S(o;(\ell,i))} \prod_{(p,q)\in T} \mu_{pq} \,. \tag{A-5}$$

The decomposition approach can be further applied to $S(o;\ \ell,i)$ at any other node j in N: the set $S(o;\ \ell,i)$ can be decomposed into several subsets, $S(o;\ \ell,i;m,j)$, by links incident to node j, $(m,j)\in I(j)$, where $S(o;\ \ell,i;m,j)$ is the subset of $S(o;\ \ell,i)$ whose element includes both links (ℓ,i) and (m,j). Hence, (A-5) is transformed to

$$Z_o = \sum_{(\ell,i)\in I(i)} \mu_{\ell i} \sum_{(m,j)\in I(j)} \mu_{mj} \sum_{T\in S(o;(\ell,i);(m,j))} \prod_{(p,q)\in T} \mu_{pq} \,, \tag{A-6}$$

where $S(o;(\ell,i);(m,j))$ is the set whose element is formed by deleting two links (ℓ,i) and (m,j) from the tree in $S(o;\ \ell,i;m,j)$. Because every tree in $S(o)$ has precisely N-1 links (that are incident to the N-1 distinct nodes), we can repeatedly apply the decomposition scheme over all nodes in N excluding node o. Consequently, we obtain

$$Z_o = \prod_{\substack{q\in N \\ q\neq o}} \left(\sum_{(p,q)\in I(q)} \mu_{pq} \right). \tag{A-7}$$

Similarly, $S(i;(i,o))$ in the definition of $Z_{i(i)}$ can be decomposed with respect to nodes: taking notice that each tree in $S(i;(i,o))$ does not include links in $I(i)$ as well as links in $I(o)$, we have

$$Z_{i(i)} = \prod_{\substack{q\in N \\ q\neq o, q\neq i}} \left(\sum_{(p,q)\in I(q)} \mu_{pq} \right). \tag{A-8}$$

From (A-7) and (A-8), $Z_{i(i)}/Z_o$ in the first term of (A-2) reduces to

$$\frac{Z_{i(i)}}{Z_o} = \frac{1}{\displaystyle\sum_{(\ell,i)\in I(i)} \mu_{\ell i}} \,. \tag{A-9}$$

We then compare $Z_{k(i)}$ and Z_o in the second term of (A-2). Note here that $Z_{k(i)}$ can be decomposed with respect to directed paths from node i to node k:

$$Z_{k(i)} = \sum_{r\in P(i,k)} \prod_{(m,n)\in r} \mu_{mn} \sum_{T\in S(i;(k,o);r)} \prod_{(p,q)\in T} \mu_{pq} \,, \tag{A-10}$$

where $P(i,k)$ is the set of directed paths from node i to node k, and $S(i;(k,o);r)$ is the set of

links whose element is formed by deleting the links in r th path in $P(i,k)$ from each pseudo-tree in $S(i; (k,o))$. The validity of this decomposition is based on the fact that each tree in $S(i; (k, o))$ has precisely a single path from node i to node k. [The reason why this fact holds is as follows: for each tree in $S(i; k,o)$, it follows immediately from the definition of $S(i; k,o)$ that there is only one path from node i to node o, and that the path should consists of the path from node i to node k and "return link" (k,o); it in turn implies that each tree in $S(i; (k,o))$ has the same single path from node i to node k because any pseudo-trees in $S(i; (k,o))$ are formed by simply deleting link (k,o) from the trees in $S(i; k,o)$.]

Furthermore, a logic similar to that in deriving the expression (A-8) can be applied to the set $S(i; (k,o); r)$ in (A-10); we split the node set N into two subsets: the set of nodes in rth path in $P(i,k)$, $N(r[i,k])$, and the set of the other nodes, $NC(r[i,k])$; then the summation with respect to $S(i; (k,o); r)$ in (A-10) can be decomposed with respect to nodes in $NC(r[i,k])$:

$$\sum_{T \in S(i;\ (k,o);\ r)} \prod_{(p,q) \in T} \mu_{pq} = \prod_{q \in NC(r[i,k])} \sum_{(p,q) \in I(q)} \mu_{pq}. \tag{A-11}$$

Accordingly, (A-10) can be written as

$$Z_{k(i)} = \sum_{r \in P(i,k)} \left(\prod_{(m,n) \in r} \mu_{mn} \right) \left(\prod_{q \in NC(r[i,k])} \sum_{(p,q) \in I(q)} \mu_{pq} \right). \tag{A-12}$$

It follows, in calculating $Z_{k(i)}/ Z_o$, that the term in the second bracket in (A-12) and the corresponding term in (A-7) mutually cancel out; that is, $Z_{k(i)}/ Z_o$ reduces to

$$\frac{Z_{k(i)}}{Z_o} = \frac{1}{\displaystyle\sum_{(\ell,k) \in I(k)} \mu_{\ell k}} \sum_{r \in P(i,k)} \prod_{(p,q) \in r} \frac{\mu_{pq}}{\displaystyle\sum_{(m,p) \in I(p)} \mu_{mp}}. \tag{A-13}$$

Substituting (A-9) and (A-13) into (A-2), we obtain (23) in Theorem 3.2.

APPENDIX B: Proof of Proposition 3.1

From the definition of $\dot{\lambda}_i(s)$ in (28), $\dot{\lambda}_i(s)$ is always zero if conditions (*a*) and (*b*) are satisfied. It also follows that the sign of $\dot{\lambda}_j(s)$ is always positive if conditions (*a*) and (*c*) are satisfied. Hence, $\dot{\lambda}_i(s) - \dot{\lambda}_j(s)$ in (27) is always negative if conditions (*a*), (*b*) and (*c*) hold. On the other hand, $\tau_j(s) - \tau_j(0)$ in (27) is always positive because the users who depart at time 0 from the origin should arrive at node *j* earlier than any users who depart at time $s > 0$ under the DUE state. Therefore, $\partial C / \partial \mu_{ij}$ is always negative (i.e., the paradox can not occur) for the network satisfying the conditions (*a*), (*b*) and (*c*). **QED.**

APPENDIX C: Proof of Proposition 3.2

Conditions (*b*) and (*c*) means that the paths from node *d* to node *j* necessarily pass through node *i*, and hence, each path in $P(d,j)$ can be obtained by adding an appropriate path in $P(i,j)$ to a path in P(*d*, *i*). Therefore, from the definition of $\dot{\lambda}_i(s)$ in (28) and condition (*a*), we have

$$\dot{\lambda}_j(s) = \alpha \; \dot{\lambda}_i(s), \tag{C-1}$$

where

$$\alpha \equiv \sum_{r \in P(i,j)} \prod_{(p,q) \in r} \frac{\mu_{pq}}{\sum_{(m,q) \in I(q)} \mu_{mq}} \; . \tag{C-2}$$

The coefficient α in (C-1) always takes the value less than one. This fact can be proved by using the concept of Markov chain networks (see, for example, Akamatsu(1996)). Consider a "reversed network" that is obtained by reversing the direction of all links in the original network $G(N, L)$, and suppose that vehicles move subject to the Markov chain rule on the reversed network with the transition probability from node *i* to node *j* defined as

$$p_{ij} = \frac{\mu_{ij}}{\sum_{(m,i) \in I(i)} \mu_{mi}}, \tag{C-3}$$

where $I(i)$ is the set of links incident to node *i*, and μ_{ij} is the capacity of link (*j*, *i*) in the *original network* (i.e., $I(i)$ and μ_{ij} are the set of links incident from node *i*, and the capacity of link (*i*, *j*)

in the *reversed network*, respectively). Then α defined by (C-2) is interpreted as the probability that vehicles that was at node *j* at an initial time will visit node *i*. Markov chain theory guarantees that the probability converges and is less than unity if the network contain no loop. Because the DUE flow pattern in a saturated network does not form loops, we can conclude that α is indeed less than unity, which means that $\dot{\lambda}_i(s) - \dot{\lambda}_j(s)$ is always positive. Therefore, $\partial C / \partial \mu_{ij}$ is always positive (i.e., the paradox always occurs) for networks that satisfy the conditions (*a*), (*b*) and (*c*). **QED.**
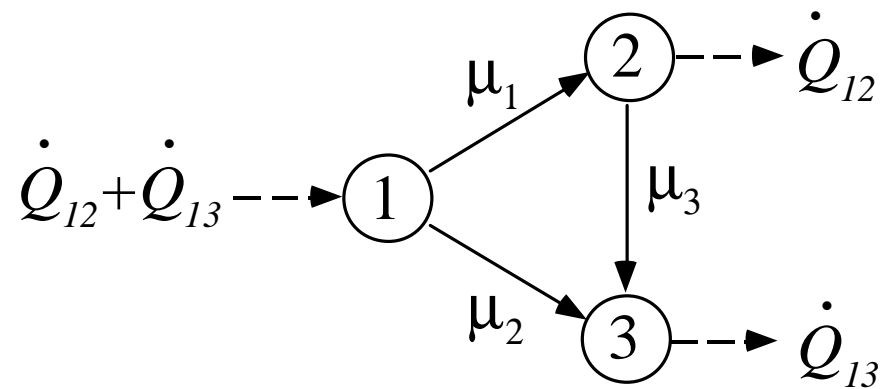
Figure 1. A Network example with a single origin and two destinations
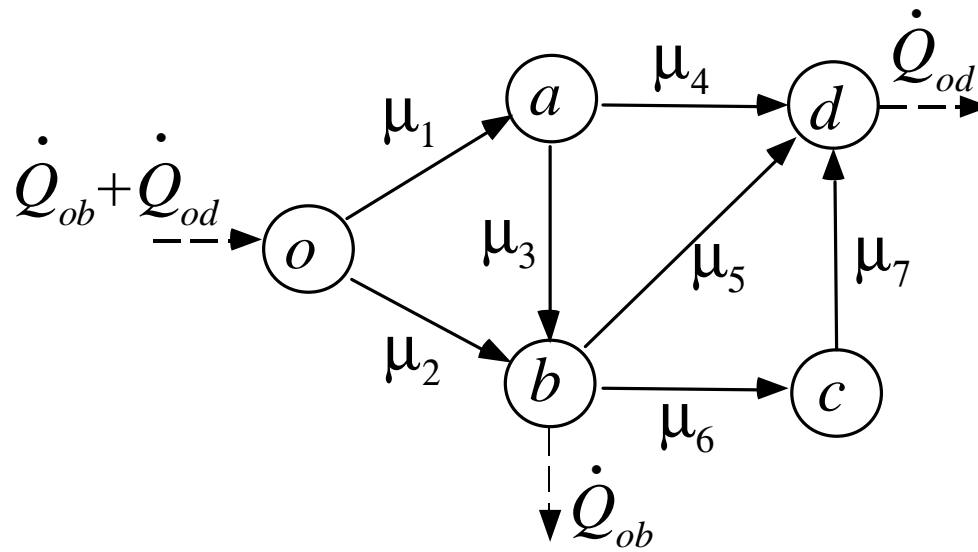
(a)   Network G

Figure 2. An example network and the corresponding graph G
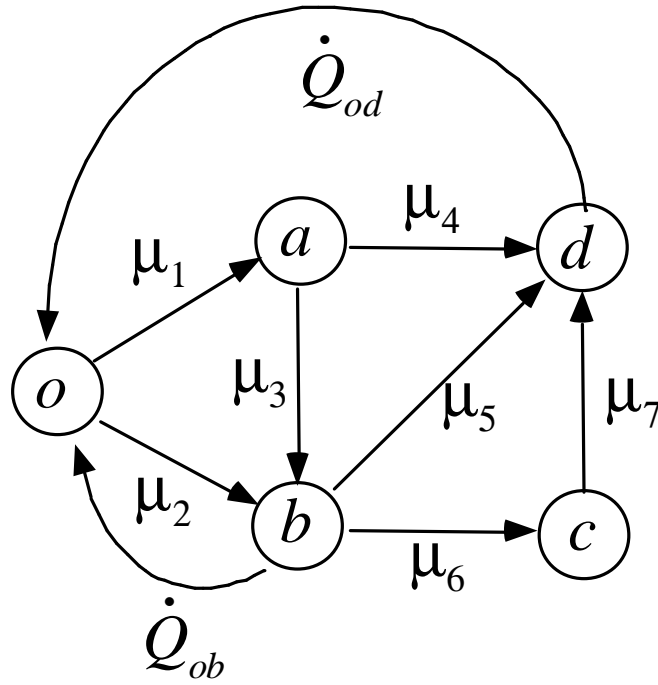
(b) Graph G

Figure 2. An example network and the corresponding graph G
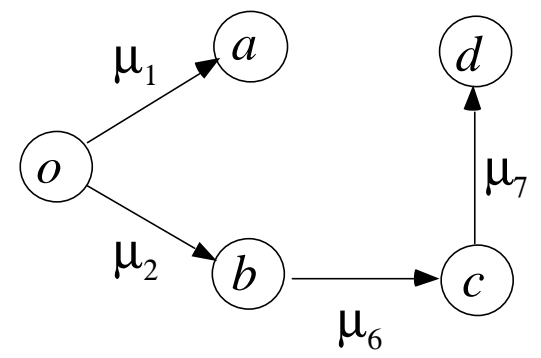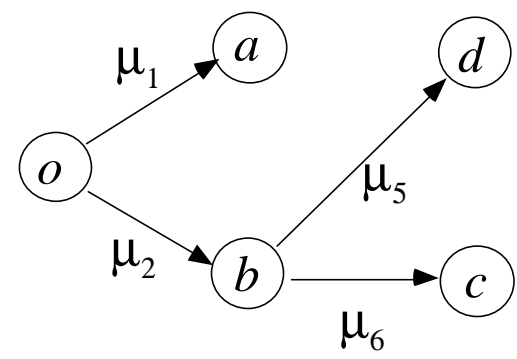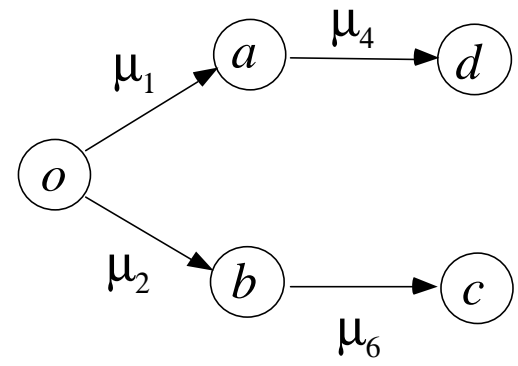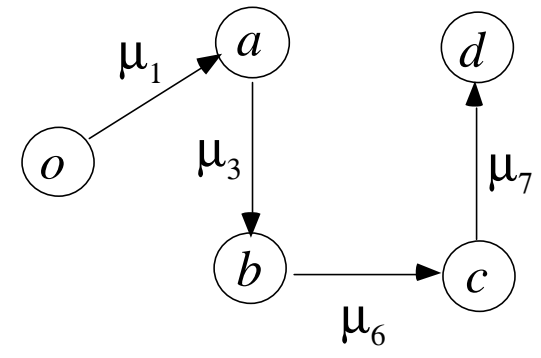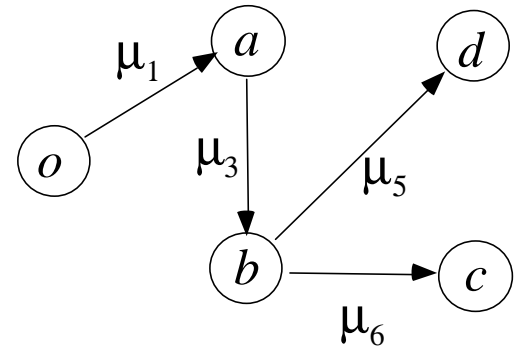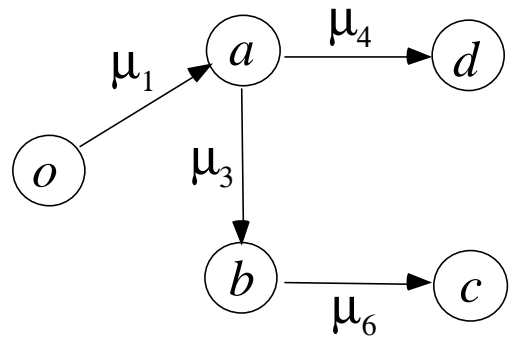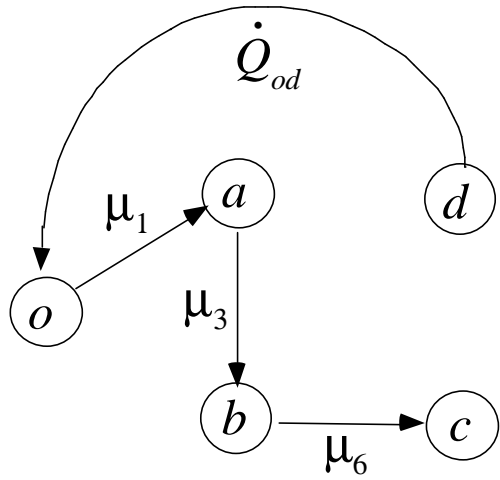
Figure 3. Directed spanning trees in S(o)
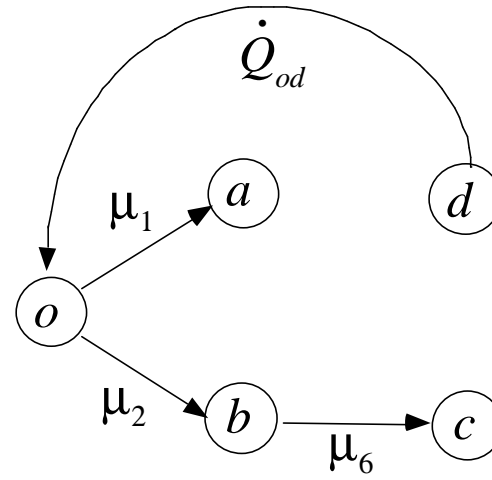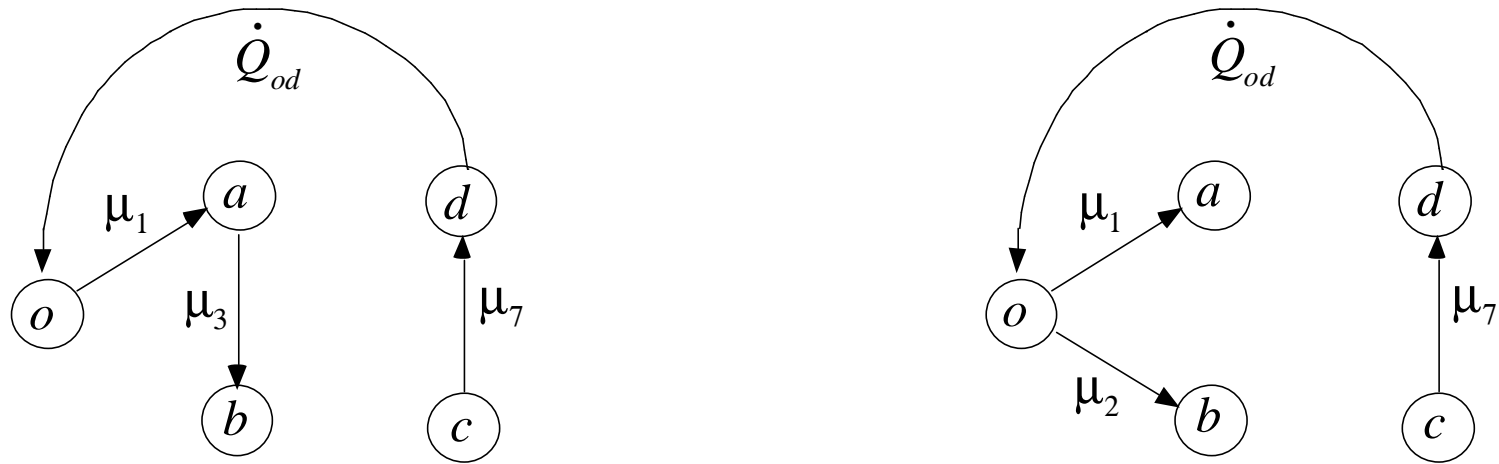
Figure 4. Directed spanning trees in S(d)

Figure 5. Directed spanning trees in S(*c*)
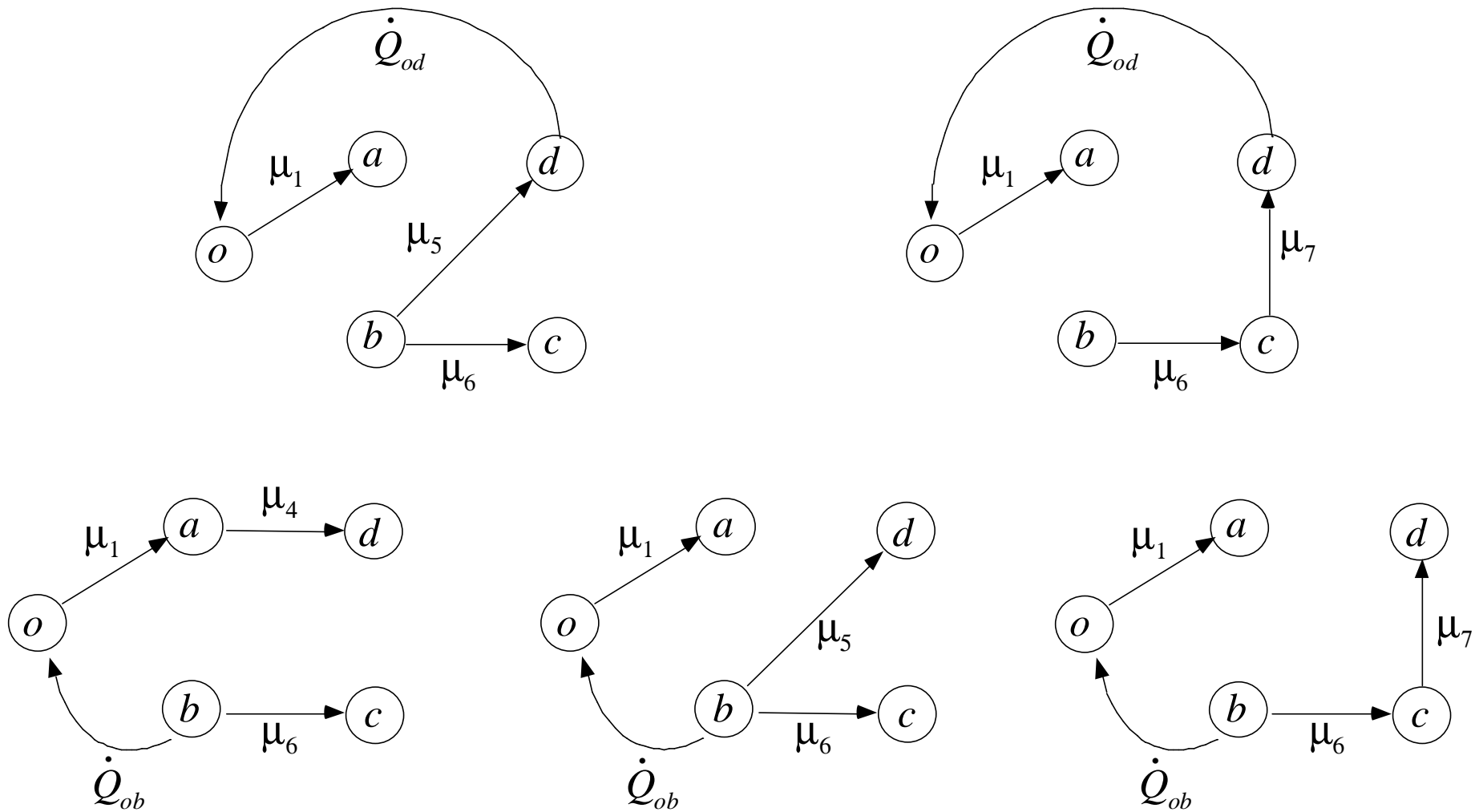
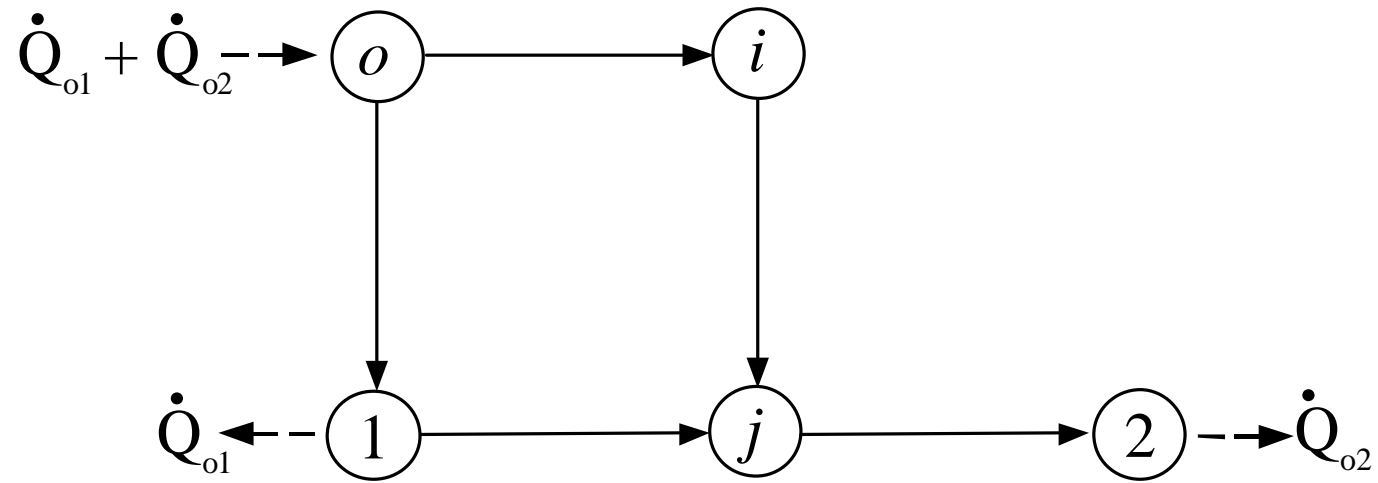Figure 6. Directed spanning trees in S(*b*)

Figure 7. A network example for Proposition 3.1 (the capacity paradox never occurs)
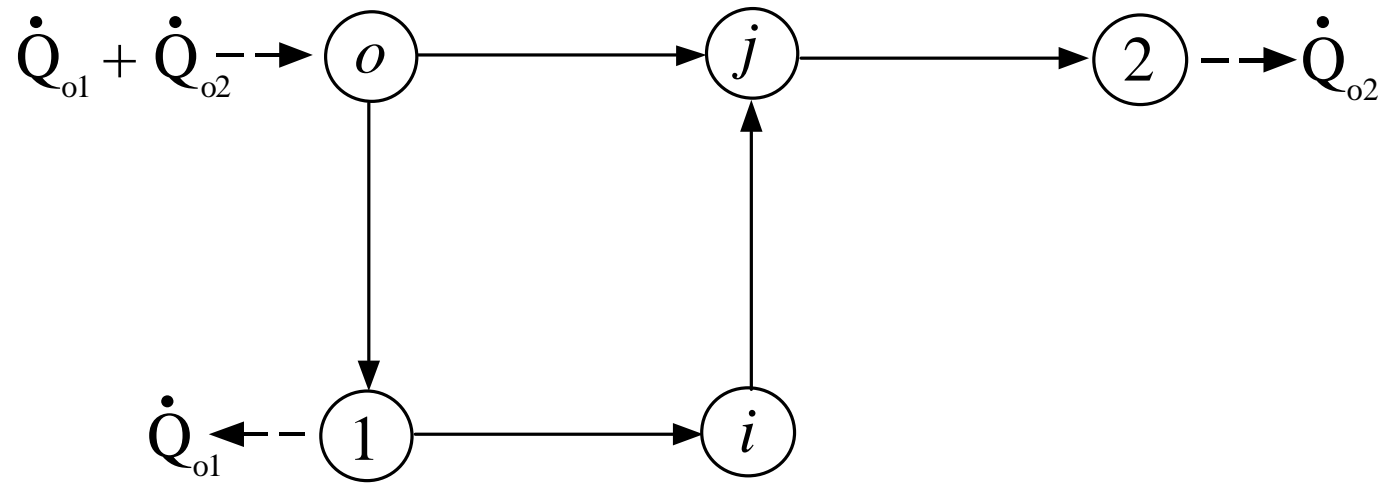
Figure 8. A network example for Proposition 3.2 (the capacity paradox always occurs)